DOCUMENT RESUME

ED 348 072 JC 880 468

AUTHOR Cohen, Don, Ed.
TITLE The AMATYC Review, Volume 10, Number 1, Fall 1988.
INSTITUTION American Mathematical Association of Two-Year Colleges.
PUB DATE 88
NOTE 93p.
AVAILABLE FROM AMATYC, Mott Community College, 1401 E. Court St., Flint, MI 48502 (3 issues free with $25.00 membership).
PUB TYPE Collected Works - Serials (022) -- Viewpoints (Opinion/Position Papers, Essays, etc.) (120)
JOURNAL CIT AMATYC Review; v10 n1 Fall 1988

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS Calculators; *College Mathematics; Computer Assisted Instruction; Computer Literacy; Geometric Concepts; Linear Programing; Mathematical Applications; *Mathematical Concepts; Mathematical Logic; *Mathematics Education; *Mathematics Instruction; Two Year Colleges

ABSTRACT

Designed as an avenue of communication for all mathematics educators concerned with the views, ideas, and experiences pertinent to two-year college teachers and students, this journal presents articles and regular features related to mathematical and pedagogical themes. This issue includes the following articles: (1) "Fractals for Freshmen? Or, Holding Infinity in the Palm of Your Hand," by Samuel F. McInroy; (2) "Shadow Prices and the Duality Principle of Linear Programming," by Sharon Kunoff; (3) "Double-Angle Triangles, Pentagons, Paradoxes and Gold," by Bill Leonard, Jan Schwartzman, and Harris Schultz; (4) "Some Applications of Almost Geometric Infinite Series," by Coreen Mett and Susan Milton; (5) "Matrices Which Are Equal to Their Own Adjoints," by David Horowitz and Nguyen Tuan Q.; (6) "How Many Final Games to Two Consecutive Championships?" by Joseph F. Stokes; (7) "The Ohio State University Calculator and Computer Precalculus Project: The Mathematics of Tomorrow Today!" by Franklin Demana and Bert K. Waits; and (8) "Using Spreadsheets to Offer a More Realistic Business Mathematics Course," by Edward D. Laughbaum. Regular features include book and software reviews. (AAZC)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED348072

THE AMATYC REVIEW

Published by the
AMERICAN MATHEMATICAL ASSOCIATION OF TWO-YEAR COLLEGES

VOLUME 10, NUMBER 1 FALL 1988

FEATURING:

Sam McInroy presents his views on how to introduce fractals into our mathematics classrooms. He is the author of

"PERMISSION TO REPRODUCE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY

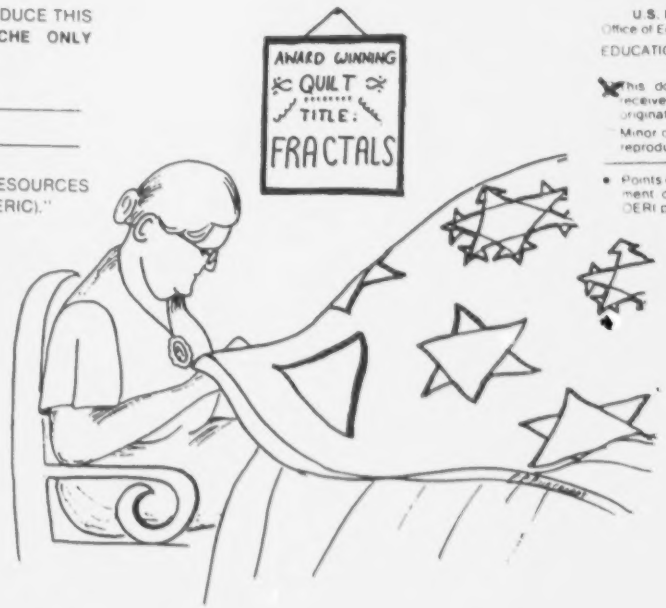
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

This document has been reproduced as received from the person or organization originating it.

Minor changes have been made to improve reproduction quality.

Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.



"Fractals for Freshmen?
Or, Holding Infinity in the Palm of your Hand"

ALSO IN THIS ISSUE:

— Shadow Prices and the Duality Principle of Linear Programming

— Some Applications of *Almost Geometric* Infinite Series

— The Ohio State University Calculator and Computer Precalculus Project: The Mathematics of Tomorrow Today!

JC 880 468

BEST COPY AVAILABLE

Focus on College Algebra

New!

by Robert Hackworth, Joseph Howland and George Schultz
St. Petersburg Junior College
Clearwater Campus

Focus on a new idea for traditional lecture/discussion teaching.

State-of-the-art knowledge of student learning influenced the design of ***Focus on College Algebra***. It graphically directs the student readers' attention to each new concept or skill in a way which will bring the reader to an understanding of the text objectives.

The need for continuous review is directly addressed in this new text. Continuous review has proved highly successful and will overcome a major difficulty with other college algebra texts — namely a tendency for students to learn each new concept as a separate idea or skill without integrating their learning.

The exercises play a major role in the learning process. Part A exercises aim at developing further understanding of the foci of their units.

Part B exercises are the traditional type of drill and practice problems associated with the units. Part C exercises are review problems and continually require the student to look again at previously studied objectives.

Occasionally there will be a Part D exercise set to expand the word problem attack skills of the readers. Word problems themselves are introduced in a unit of Chapter 1 and are included in every Part C exercise thereafter.

Competency in College Mathematics, 4th Ed.

by Jack C. Gill
and Robert Blitzer
Miami-Dade Community College

"Many states are now mandating that students achieve a mastery of certain mathematics concepts. As these mandates have developed there is an increasing consensus toward a mastery of many skills and concepts associated with courses in traditional liberal arts mathematics and finite mathematics. This text meets the political mandates of our times, but also maintains the solid academic integrity of our profession. We proudly recommend its mathematics, its instruction, and its academic standing." *from the preface*

- Covers all traditional liberal arts math topics and includes the most extensive logic section of any book of this type.
- Review exercises are included in each exercise section.
- Problem sets occur after just a few concepts are introduced. This helps the student master concepts sequentially and in relatively small increments. All answers are provided to insure maximum feedback.
- Produced entirely with electronic publishing technology to provide visual clarity and readability at low cost.

College Math Review, 3rd Ed.

by Robert Blitzer
and Jack C. Gill
Miami-Dade Community College

- Topics include Sets & Logic, Probability & Statistics, Algebra & Number Theory, Informal Geometry and Basic Arithmetic.
- New revised and expanded edition includes chapter summaries and sample examinations.
- Brief, expository format, hundreds of illustrative problems, and over 1000 class-tested homework problems.

H&H Publishing Company, Inc.

2165 Sunnydale Blvd., Suite N, Clearwater, FL 34625-1211, (813) 442-7760

The Official Journal of the

American Mathematical Association of Two-Year Colleges

PURPOSE

AMATYC remains committed to its original purposes:

- To provide a national forum for the exchange of ideas
- To further develop and improve the mathematics education of students of two-year colleges
- To coordinate activities of affiliated organizations on a national level
- To promote the professional development and welfare of its members

The AMATYC Review provides an avenue of communications for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

SUBMISSION OF MANUSCRIPTS

MANUSCRIPTS MUST CONFORM TO THE FOLLOWING FORMAT:

They must be typewritten on one side and double-spaced on 8½" x 11" paper. Five copies of each manuscript, which has not been published before, should be submitted to the editors. If the manuscript contains any photographs, they should be black and white glossy prints. The first page of the manuscript should contain the article's title, all the authors' names and their institutional affiliation along with the name and address of the person to whom all correspondence should be mailed. Authors are advised to consult the *Publication Manual of the American Psychological Association* (3rd ed.). A guideline for authors is available from the editor.

Manuscripts should be sent to: Don Cohen, Editor, *The AMATYC Review*, SUNY College of Agriculture and Technology, Cobleskill, NY 12043. Any manuscript not accepted for publication will be returned.

ADVERTISING

For information concerning advertising rates and schedules, write to Eleanor Young, Sinclair Community College, Dayton, OH 45402 or telephone (513) 226-2789.

STAFF

Editor: Don Cohen, SUNY Ag-Tech College, Cobleskill, NY 12043, (518) 234-5116

Production: Roy Cameron, SUNY Ag-Tech College, Cobleskill, NY 12043, (518) 234-5351

Advertising: Eleanor Young, Sinclair Community College, Dayton, OH 45402

Artwork: Robert Burghardt, Rockland Community College, Suffern, NY 10901

PUBLICATION

The AMATYC Review is published twice a year in the Fall and Spring.

ISSN 0740-8404

Brooks/Cole
Publishing Company

Offering a wide range of texts for precalculus, applied calculus, and beyond

NEW FOR 1989

Mathematics for Calculus
James Stewart, Lothar Redlin, Saleem Watson

Applied Calculus, Second Edition
by Claudia Taylor and Lawrence Gilligan

Calculus with Applications
by Claudia Taylor and Lawrence Gilligan

Elements of Calculus, Second Edition
by G.D. Allen, William Perry, and Charles K. Chui

Linear Algebra with Applications, Third Edition
by Jeanne L. Agnew and Robert C. Knapp

Ordinary Differential Equations with Applications
by Bernard J. Rice and Jerry D. Strange

To request a complimentary copy of any of these texts, write on department letterhead to:

Department 89208A
Brooks/Cole Publishing Company
511 Forest Lodge Rd.
Pacific Grove, CA 93950

The AMATYC Review **Fall, 1988** **Volume 10, Number 1**

Editorial ..... p. 5
Letter to the Editor ..... p. 8

VIEWPOINT

Fractals for Freshmen? Or, Holding Infinity in the Palm of your Hand ..... p. 10
by Samuel F. McInroy

MATHEMATICAL EXPOSITION

Shadow Prices and the Duality Principle of Linear Programming ..... p. 17
by Sharon Kunoff
Double-Angle Triangles, Pentagons, Paradoxes and Gold ..... p. 26
by Bill Leonard, Jan Schwartzman, and Harris Shultz
Some Applications of *Almost Geometric* Infinite Series ..... p. 33
by Coreen Mett and Susan Milton

SHORT COMMUNICATIONS

Matrices Which are Equal to Their Own Adjoints ..... p. 41
by David Horowitz and Nguyen Tuan Q.
How Many Final Games to Two Consecutive Championships? ..... p. 43
by Joseph F. Stokes

MATHEMATICS EDUCATION

The Ohio State University Calculator and Computer Precalculus Project:
The Mathematics of Tomorrow Today! ..... p. 46
by Franklin Demana and Bert K. Waits
Using Spreadsheets to Offer a More Realistic Business Mathematics Course ..... p. 57
by Edward D. Laughbaum

REGULAR FEATURES

Educational Reflections ..... p. 63
by Gregory D. Foley
Pandora's Rectangular Parallelipiped ..... p. 68
Edited by Leslie R. Tanner
Book Reviews ..... p. 71
Edited by John J. Edgell
Problem Section ..... p. 74
Edited by Michael W. Ecker
Computer Software Section ..... p. 79
Edited by Shoa Mah
Mathematical Trivia ..... p. 80
Edited by Ralph Selig

MATHEMATICAL TRIVIA (MT)

Edited by Ralph Selig
Fairleigh Dickinson University
Hackensack, NJ 07601

Below and in other places in this journal are some pieces of mathematical trivia that will hopefully challenge and entertain our readers. Contributions of trivia items such as not so famous quotes, historical anecdotes and short puzzles may be submitted to the editor.

MT 1. Find the next two numbers in the following sequence:

3, 7, 31, 127, _____, _____

Editor's Comments

by Don Cohen

The Helpful Mathematics Educator

I am sure that all educators have the feeling that helping others to learn can be a personally rewarding experience. Perhaps this is why we became educators in the first place. Let me cite two out of the ordinary types of situations that I found to be especially rewarding.

Helping Colleagues

During my 22 years of teaching mathematics at the two year college level, several colleagues have asked me to help them with their research (usually as part of their doctoral dissertations) either by answering questions on statistical matters or by "coercing" my students to be subjects in their experiments. I am always happy to help. In addition to these ordinary requests, I have been asked by two colleagues to get very deeply involved in the development of their dissertations, and it is these latter requests that I would like to specifically mention.

The first request was from a doctoral student at Nova University in Fort Lauderdale, Florida. I'm sure that you can imagine the difficulties inherent in writing a dissertation in Cobleskill, New York while enrolled at a university in Florida. I served as the student's "unofficial" mentor helping her with her research statistics. Perhaps more importantly, I made myself available to read and discuss the preliminary drafts of her dissertation prior to her sending them to her advisor. It was very satisfying for me to congratulate Sylvia when she received her degree.

The second request was by a doctoral student in mathematics education at the State University of New York at Albany who will soon complete his degree requirements. In this case, I was asked to serve on the student's dissertation committee. In my role as committee member, I read progressive versions of a dissertation on the teaching of calculus and I have made several suggestions to the student individually and during committee meetings. I feel that I have been an integral part of the process of developing the dissertation, because I am the only member of the committee who is currently directly involved in teaching calculus. Norm is currently putting the finishing touches to his dissertation and, again, it will be very satisfying to congratulate him when he receives his degree.

I am not commenting on my outside of class activities to make myself appear to be the "Mr. Goodwrench" of the education profession. These comments are simply meant to suggest that when you help your colleagues, you may be helping yourself to a very rewarding educational experience.

Helping Special Students

What do you know about educating students with learning disabilities? I used to think that the words "learning disability" were simply fancy words meaning "lazy" or "low ability;" and, if I yelled a little louder, I could get these students to learn. Then I went to a one-day seminar sponsored by our faculty union and the State University of New York that convinced me that there is much more that I should know about the very real problems associated with educating students who have learning disabilities.

> A learning disability is a permanent disorder which affects the manner in which individuals receive, process, retain, and/or express information (Garret and Welch, 1987, p. 3).

It should be emphasized that no amount of yelling is going to help. Learning disabilities are permanent handicaps. Furthermore, the seminar leaders reminded us that it is the responsibility of teaching faculty to provide optimal learning environments for all students in their classes including those with learning disbilities.

I know that many readers are now saying to themselves, "Wait a minute. I have a large teaching load. I cannot provide individualized instruction." My response is simple. "Find out what you can potentially do to help all students learn with maximal efficiency and then do whatever you can comfortably do to help."

Let me cite the first step that I have made in this direction. After returning from the seminar, I simply announced to my classes that students with diagnosed learning disabilities who have special needs are invited to meet with me individually in my office. Notice that I did not ask them to publicly identify themselves. I had one student accept my invitation. It seems that Kathy became very nervous when taking a test in a large classroom setting with a definite time limit. Since speed in doing elementary algebra is not a major concern, I had no problem with allowing her to take tests individually in my office without a specific time constraint. Kathy always finished within an hour anyway. By the way, I did not receive any "grief" from other students in Kathy's class who knew about her special situation. Simple steps can go a long way toward providing an effective learning environment for a diverse student population.

An Invitation to Contribute to a Special Theme Section

As a way of getting our readers more involved with contributing manuscripts to *The Review*, I would like to initiate a theme section in our Spring, 1988 issue. Our initial theme topic will be "A Most Appreciated Mathematics Teacher." You are invited to contribute brief (2-3 double-spaced pages) manuscripts that outline the careers of collegiate-level mathematics faculty and explain how these individuals greatly influenced your teaching careers. The purpose of this theme section is two fold: First, your article will give you the opportunity to publicly show your appreciation for the efforts of an individual whom you may have wanted to thank for many years. And, perhaps more importantly, you have the opportunity to explain to others the attributes of a person whom the readers may want to adopt as a role model.

I will need to receive these manuscripts by November 7, 1988 at the very latest so that they may be included in our Spring, 1989 issue. Please include a black and white picture of your subject with your manuscript. If you have any topic ideas for future theme sections, please send them to the editor.

A Welcome to Our New Production Manager

Roy Cameron, a colleague in the Mathematics Department at SUNY Cobleskill, was appointed to the post of production manager for *The Review* in November, 1988. Roy is a graduate of SUNY at Albany and he began his teaching career at Cobleskill in 1968. He is a past president of the New York State Mathematics Association of Two Year Colleges. Welcome to our staff.

At this same time, I would like to thank our former production manager, Paul Dudenhefer, of the State Technical Institute at Memphis for his dedicated efforts. Paul was initially appointed as our production manager in 1983.

References

Garret, M. K. & Welch, E. L. (1987). *Serving the student with a learning disability: A manual for SUNY faculty and professionals.* Albany, NY: SUNY Affirmative Action Office.

The Formula For Success
—MATH TEXTS FROM SAUNDERS!

New!

FUNDAMENTALS OF MATH
ELEMENTARY ALGEBRA
INTERMEDIATE ALGEBRA

All by Denny Burzynski and Wade Ellis, both of West Valley College

Used separately or in sequence, this new series of texts will take your students from fundamentals of mathematics to basic algebra to an intensive study of intermediate algebraic topics. Each text uses "Sample Sets" that show students how to work problems and "Practice Sets" to develop students' problem-solving abilities.

FUNDAMENTALS OF MATH ISBN 0-03-063901-8 550 pp. paper
ELEMENTARY ALGEBRA ISBN 0-03-063906-9 608 pp. paper
INTERMEDIATE ALGEBRA ISBN 0-03-063903-4 604 pp. paper

The Barker/Rogers/Van Dyke Series: Revised and Available Now!

Jack Barker, James Rogers, and James Van Dyke, all of Portland Community College

ARITHMETIC, 4/E

Problem-solving skills and their applications are emphasized in this comprehensive survey of arithmetic concepts. ARITHMETIC offers student-oriented explanations with an adult approach.

ISBN 0-03-007033-3 596 pp. paper

BASIC ALGEBRA, 2/E

This work text on elementary algebra offers an approach flexible enough to be adaptable for either a classroom lecture format or individualized instruction.

ISBN 0-03-007052-X 788 pp. paper

INTERMEDIATE ALGEBRA, 2/E

Problem-solving procedures are clearly explained in this thorough presentation of intermediate algebra which focuses on manipulative skills and their applications.

ISBN 0-03-007068-6 788 pp. paper

New!

COLLEGE ALGEBRA
TRIGONOMETRY
ALGEBRA AND TRIGONOMETRY

All by Stanley I. Grossman, University of Montana

For your college algebra and trigonometry courses, this series of three texts stresses real-life applications, exercises and lively historical anecdotes to help your students more readily absorb and truly enjoy the material.

COLLEGE ALGEBRA ISBN 0-03-007089-9 480 pp. hardcover
TRIGONOMETRY ISBN 0-03-007103-8 384 pp. hardcover
ALGEBRA AND TRIGONOMETRY ISBN 0-03-007129-1 512 pp. hardcover

New!

FINITE MATHEMATICS WITH APPLICATIONS
CALCULUS WITH APPLICATIONS
FINITE MATHEMATICS WITH CALCULUS: AN APPLIED APPROACH

All by Raymond F. Coughlin and David E. Zitarelli, both of Temple University

Each text in this series emphasizes application of concepts through the use of case studies in each chapter. Each case study is accompanied by examples and exercises related directly to that case. Chapter summaries, worked examples and graded problem sections aid in students' understanding of the material.

FINITE MATHEMATICS ISBN 0-03-011292-3 432 pp. hardcover
CALCULUS WITH APPS ISBN 0-03-011283-4 640 pp. hardcover
FINITE MATHEMATICS WITH CALCULUS ISBN 0-03-011274-5 800 pp. hardcover

Revised and Available Now!

CALCULUS, 2/E

Dennis D. Berkey, Boston University

ISBN 0-03-008989-2 1,152 pp. hardcover

DIFFERENTIAL EQUATIONS: A FIRST COURSE, 2/E

Martin M. Guterman and Zbigniew H. Nitecki, both of Tufts University

ISBN 0-03-059617-0 656 pp. hardcover

STATISTICS AND PROBABILITY IN MODERN LIFE, 4/E

Joseph Newmark, The College of Staten Island, CUNY

ISBN 0-03-008367-2 680 pp. hardcover

How to Order: For examination copies, please contact your local Saunders College Publishing sales representative or write on your college letterhead to: Mathematics Marketing Manager, Dept 544, Saunders College Publishing, Box 181, Lavallette, NJ 08735. Include your course title, enrollment, and text currently in use. To expedite shipping, please include the ISBN (International Standard Book Number) for each item requested.

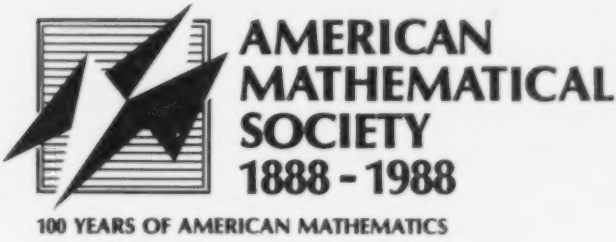
SAUNDERS

1888 1988

A Century of Publishing for the Future

Saunders College Publishing, The Curtis Center
Independence Square West, Philadelphia, PA 19106

Letter to the Editor



Dear Editor,

This year, the American Mathematical Society celebrated its 100th anniversary. The last hundred years have witnessed an unprecedented growth in mathematical research and will surely be known as one of the golden ages of mathematical history. Throughout the past century, the Society has pursued through many diverse activities its aim of furthering mathematical scholarship and research. In the past three decades we have seen the two-year college mathematics community assume a major role in the educating of students in undergraduate mathematics, and we join the rest of the mathematical community in congratulating the American Mathematical Society on this anniversary.

Three noteworthy American mathematicians who shaped the course of worldwide mathematics in the last hundred years are George David Birkhoff, of Harvard University, who distinguished himself by proving Poincare's "last geometric theorem", Norbert Wiener, of M.I.T., and John von Neumann of Princeton, who set the tone for much of American mathematical research in the 1930's.

Recently, the United States has taken the lead in the development of the computer, information sciences, and of modern applications of mathematics to real world problems. The past few years have been particularly exciting for all mathematicians in both the theoretical and applied areas. Some famous old problems have been solved, and an entirely unexpected phenomenon has been discovered in four dimensional space. On the practical side, a new method of solving large linear programming problems has been discovered which promises to greatly reduce computing time and to bring much larger problems within the range of our computers.

In just 100 years, the American mathematical community has been established as a force within the nation and the American mathematical presence has firmly established itself globally.

In its next 100 years in America, mathematics will increasingly provide the reach and vital language of our technological society, helping the United States to function as a constructive world leader.

Sincerely,
Karl J. Smith
President, AMATYC

From W. H. Freeman and Company

Leading mathematicians nationwide were asked,
"What would you teach liberal arts students
if they took only one college mathematics course?"
Their answer led to:

FOR ALL PRACTICAL PURPOSES
Introduction to Contemporary Mathematics
Consortium for Mathematics and Its Applications, Inc.

FOR ALL PRACTICAL PURPOSES is a new textbook that adopts a unique, applications-oriented approach to convey the vitality of mathematics and the central role it plays in today's increasingly technological society. The solid decision-making skills developed in this applied setting will serve your students far beyond the confines of the classroom—in every part of their lives for the rest of their lives.

FOR ALL PRACTICAL PURPOSES can either stand alone or be used in conjunction with the PBS telecourse of the same name, available from the Annenberg/CPB Project. For information on the telecourse, call **1-800-LEARNER.**

450 pages, 360 illustrations Cloth ISBN 0-7167-1830-8
Ancillaries: Instructor's Manual and Telecourse Guide for Students

GEOMETRY Second Edition
Harold R. Jacobs

Thoroughly rewritten, the Second Edition perfects an already acclaimed approach to the teaching of introductory geometry. Extensively class-tested, GEOMETRY now contains almost twice as many exercises as the previous edition; algebra review questions at the end of selected chapters; midterm and final review tests; a glossary; and a complete support system for instructors.

688 pages, 1962 illustrations Cloth ISBN 0-7167-1745-X
Ancillaries: Teacher's Resource Guide with Solutions,
Test Masters, and Transparency Masters

THE MATHEMATICAL TOURIST
Snapshots of Modern Mathematics
Ivars Peterson

Science journalist Ivars Peterson takes us on an informal and entertaining journey from fractals, computer graphics, and code breaking to the search for the largest prime number and the realm of higher dimensions. THE MATHEMATICAL TOURIST is an imaginative tour through the fascinating world of modern mathematics.

240 pages, 16 pages color, 119 illustrations Cloth: ISBN 0-7167-1953-5
(Sorry, no complimentary copies available on this title.)

Send your request for a complimentary copy on school letterhead, stating course title, number, enrollment, and current text in use to:
Marketing Department
W. H. Freeman and Company
41 Madison Avenue, New York, NY 10010

W. H. Freeman and Company The book publishing arm of *Scientific American*

VIEWPOINT

Editor's Note: On April 21, 1988 I had the pleasure of attending a ceremony to award the honorary degree of Doctor of Science to Benoit B. Mandelbrot at the State University of New York at Albany. At the ceremony the following citation was read:

Mandelbrot Citation

"Science has long pursued the study of order and regularity and has developed techniques to handle randomness.
You, BENOIT MANDELBROT, have perceived a use for geometry which has allowed us to cope with the realm between order and complete chaos, a perception that has permitted the recognition of order where it was not perceived before.
You recognized that most disorder in nature contained an underlying structure which you described with fractal geometry, the geometry of fractional, or non-integer, dimensions.
This advance marked a major intellectual revolution of our time, a paradigm change that will forever alter how humankind views the world.
Not only did you perceive this paradigm, you were the missionary who made it happen.
Now your vision is being applied to the search for knowledge and understanding in virtually all areas.
The poet, Edna St. Vincent Millay wrote that "Euclid alone has looked on beauty bare."
You too share this distinction and that vision.
For your contribution in bringing theory to practice, the State University of New York confers on you, BENOIT MANDELBROT, the degree DOCTOR OF SCIENCE."

In the article that follows, Sam McInroy presents his views on how Mandelbrot's work can be included in our classrooms.

Fractals for Freshmen
Or, Holding Infinity in the Palm of your Hand

by Samuel F. McInroy
Corning Community College
Corning, NY 14830

Sam McInroy has taught at CCC since 1964. He is a past president of NYSMATYC and a past Northeast Vice President of AMATYC. His interests include math puzzles, teaching, dart throwing, broccoli stir frying and 33mm photography.

What do the San Andreas Fault, ferns, protein molecules, broccoli, the Chambered Nautilus and flame surfaces have in common? Each has a component which may be interpreted using fractal analysis. Fractals possess the property of self-similarity: any part

is similar to the whole. A fern is an excellent example (Figure 1). Any leaf has the same shape as the entire fern. Furthermore, each leaf has even smaller parts similar to the original, and so on, *ad infinitum*.

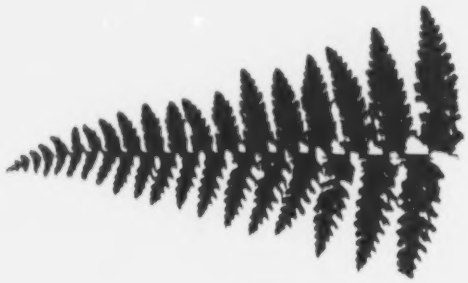


Figure 1

In my opinion, fractal analysis will assume an increasingly important role in future math/science/engineering curricula. Therefore, I believe it deserves an introduction during the first year of calculus. To this end, I propose an interesting fractal problem to my students after they have studied improper integrals and infinite series. I give them the aforementioned fern introduction and then introduce them to a "Koch" snowflake. We start with an equilateral triangle of side "s" (Figure 2). A star of David is next formed by constructing three smaller equilateral triangles using, as base, the middle thirds of each side of the original (Figure 3). Then, using a similar process with each of these 12 segments, we obtain Figure 4.

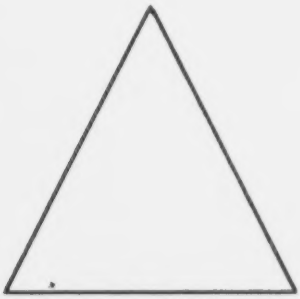


Figure 2

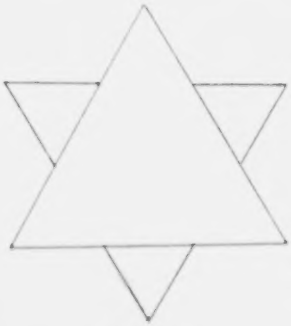


Figure 3

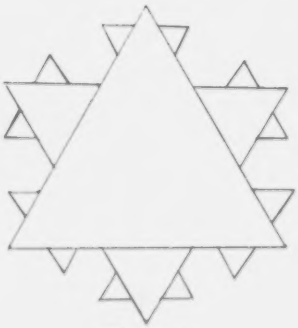


Figure 4

Students are now asked to calculate the perimeter and area of the figure through each of these three stages, and then for a fourth, fifth, and nth stage. If we concentrate on 1/3 of the perimeter at each stage, a pattern is quickly formed. At stage one, P/3 is s. At stage 2, P/3 is composed of 4 line segments, each of length s/3. In turn, at stage 3, each of these four line segments has been replaced by four more segments of length 1/3 of each stage 2 segment and thus 1/9 of stage 1. This infinitely replicating process of replacing each segment with four smaller ones, each one third the length of the previous, causes successive perimeters to grow at a 4/3 rate. In summary, 1/3 of the perimeter grows from s to (4/3)s, $(4/3)^2$s, $(4/3)^3$s,..., $(4/3)^{n-1}$s. Thus as n gets arbitrarily large, so also does the perimeter.

Observe that beginning with the 2nd stage, the number of new triangles in order is {3, 12, 48,...}. By factoring a 3 out we can look at {1, 4, 16,...}. This is equivalent geometrically to observing that, at this second stage (Figure 5), the number of triangles added is three times the number added from P_1 to P_2.

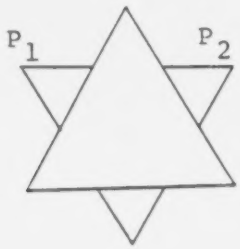


Figure 5

There are four line segments from P_1 to P_2. Due to the self-similarity of this fractal, each of these 4 segments, when subtracted to the next stage of adding an equilateral triangle with base equal to its middle third, will, upon 3X magnification, look exactly like the 3nd stage route from P_1 to P_2.

Therefore, as we proceed through stages *ad infinitum*, each time we create four times as many new triangles as before, and so, the total number of triangles added in each stage beginning with the second is

$$\{3,(3)(4),(3)(4^2),(3)(4^3),\ldots,(3)(4^{n-1}),\ldots\}. \qquad (1)$$

Note that the perimeter calculations involved *replacing* segments with four smaller segments. Area calculations will involve *adding* the areas of the successively smaller triangles to the original area. Since self-similarity is maintained, and since both the base and the height of the successively smaller triangles added at each stage are 1/3 of those generated at the previous stage, we can establish an area pattern:

> If the area of the original triangle (Figure 2) is A_1, then the area of each of the three smaller triangles in Figure 3 is $A_1/9$.

Each of the yet smaller triangles in Figure 4 has an area 1/9 of those added at stage 2 and therefore 1/81 of A_1. As a result, our middle thirds construction has caused each of the increasingly smaller triangles to have areas of

$$\left\{ \frac{A_1}{3^2}, \frac{A_1}{3^4}, \frac{A_1}{3^6}, \frac{A_1}{3^8}, \ldots, \frac{A_1}{3^{2n}}, \ldots \right\} \qquad (2)$$

Multiplying the elements of (1) term by term with the elements of (2) yields the following sequence which gives us the areas added at each stage:

$$\left\{ \frac{A_1}{3}, \frac{4A_1}{3^3}, \frac{4^2A_1}{3^5}, \ldots, \frac{4^{n-1}A_1}{3^{2n-1}}, \ldots \right\}.$$

As a result, our total area, if we continue this "infinitely replicating process" is the original area A_1 plus the expression

$$\sum_{n=1}^{\infty} \frac{4^{n-1}}{3^{2n-1}} A_1 .$$

Note that this latter term is a geometric series which converges to $\frac{3}{5} A_1$. It follows that our total area is $A_1 + \frac{3}{5} A_1 = \frac{8}{5} A_1$.

We have demonstrated that our Koch snowflake has a finite area but an infinite perimeter! This realization corresponds with the first time students were confronted with

$$\int_1^{\infty} \frac{dx}{x^2}$$

which again represents a finite area with an infinite perimeter. Students may now wish to consider how much paint would be needed to paint the interior of a "unit" snowflake . . . and to consider how much paint would be needed to "trim" the border with another color.

Another fascinating quality of our snowflake is its dimension. Students are used to identifying $3^1=3$, $3^2=9$, $3^3=27$ as having a linear, square, or cubic geometrical interpretation. Thus when $3^D=3$ we know we have dimension 1. When $3^D=9$, we have dimension 2, etc. In our snowflake example, $3^D=4$ (since three segments were replaced by four) thus $D=\log 4 / \log 3 = 1.262$, a fractional dimension. Of course, if one replaces the middle thirds with something like ⎍ instead of ⩓, then our snowflake has a different area and a dimension $\log 5 / \log 3$ ($3^D=5$).

Benoit Mandelbrot, generally acknowledged as the "father of fractals", argues that this type of fractal should be introduced when one presents the concept of derivative as it is an excellent example of a continuous curve which is nowhere differentiable.

Nature has a fondness for self-similarity. The chambered nautilus (Figure 6), beaver tooth and elephants' tusks are examples of the logarithmic spiral. As they grow, they change their size but not their shape. Magnification of a part of the spiral will yield a segment exactly like a different segment of the original. This topic may be introduced using Fibonacci and swirling golden rectangles. The very definition of golden proportion ensures self-similarity.

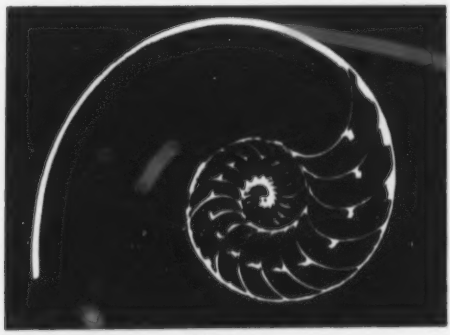


Figure 6

Since this article is an elementary introduction to fractals, I have included an extensive bibliography for readers wishing more information. Primary sources for this article are preceded with an asterisk.

In conclusion, we can turn to that supreme artist, William Blake (C1803), for an eloquent description of the mathematical poetry that is fractal analysis:

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

Bibliography

Books

Gleick, J. (1987). *Chaos.* New York: Viking Press.

*Mandelbrot, B. (1983). *The fractal geometry of nature.* New York: W. H. Freeman.

*Thompson, D. W. (1961). *On growth and form.* New York: Cambridge University Press.

*Zippin, L. (1975). *Uses of infinity.* Washington, D.C.: Mathematical Association of America.

Articles

Bahn, C. R. (1986). Musical fractals. *Byte, 11,* 185.

*Barcellos, A. (1984). The fractal geometry of Mandelbrot. *College Mathematics Journal, 15,* 98-119.

Davis, M. (1984). Interview: Benoit B. Mandelbrot. *Omni, 6,* 64-66.

Dewdney, A. K. (1985). A computer microscope zooms in for a look at the most complex object in mathematics. *Scientific American, 253,* 16-24.

*Egsgard, John C. (1988). An interesting introduction to sequences and series. *Mathematics Teacher, 81,* 108-111.

Gleick, J. (1985, December 8). The man who reshaped geometry. *New York Times Magazine.* pp. 64-65.

Kadanoff, L. P. (1986). Fractals: where's the physics? *Physics Today, 39,* 6-7.

Land, B. R. (1986). Dragon. *Byte, 11,* 137-138.

*Lewis, M. & Rees, D. C. (1985). Fractal surfaces of proteins. *Science, 230,* 1163-1165.

Lewis, M. & Rees, D. C. (1985). The texture of clinging proteins. *Science News, 128,* 377.

Robinson, A. L. (1985). Fractal fingers in viscous fluids. *Science, 228,* 1077-1080.

*Sander, L. M. (1987). Fractal growth. *Science American, 256,* 94-100.

Schroeder, P. B. (1986). Plotting the Mandelbrot set. *Byte, 11,* 207-210.

Sorensen, P. R. (1984). Fractals. *Byte, 9,* 157-158.

Staff. (1986). A fractal universe? *Science News, 129,* 217.

Staff. (1987). Unmasking chaos. *Scientific American, 256,* 61-62.

Weisburd, S. (1985). Fractals, fractures and faults. *Science News, 127,* 279.

Poems

*Blake, W. (c1803). Augaries of innocence. *Pickering Manuscript.*

MT 2. Can you name Euclid's Five Postulates in Geometry?

We're proud to announce. . .

The Wm. C. Brown Mathematics Educational Grants

Consisting of:

*one $750 departmental grant
*one $500 departmental grant
*one $250 departmental grant

To be sponsored by Wm. C. Brown Publishers at the 1988 AMATYC Convention. These grants are in appreciation and thanks to you, our valued customers, for an outstanding year.

For more details contact:

Wm. C. Brown Publishers

2460 Kerper Blvd., P.O. Box 539
Dubuque, IA 52001 (319)589-2963

The Prindle, Weber & Schmidt Series in Mathematics...

Your Single Best Source For Developmental Mathematics Titles

*** NEW FOR 1989**

BEGINNING ALGEBRA, Fourth Edition
Alfonse Gobran, Los Angeles Harbor College

INTERMEDIATE ALGEBRA, Fourth Edition
Alfonse Gobran, Los Angeles Harbor College

ALGEBRA FOR COLLEGE STUDENTS
James W. Hall, Parkland College

***ELEMENTARY ALGEBRA FOR COLLEGE STUDENTS, Third Edition**
Jerome E. Kaufmann

*** INTERMEDIATE ALGEBRA FOR COLLEGE STUDENTS, Third Edition**
Jerome E. Kaufmann

*** ALGEBRA FOR COLLEGE STUDENTS, Third Edition**
Jerome E. Kaufmann

*** ALGEBRA WITH TRIGONOMETRY FOR COLLEGE STUDENTS, Second Edition**
Jerome E. Kaufmann

BASIC MATHEMATICS, Second Edition
Rosanne Proga

*** ARITHMETIC AND ALGEBRA, Second Edition**
Rosanne Proga

INTRODUCTORY ALGEBRA
Wayne Strnad, Triton College

ARITHMETIC
Martha Wood and Peggy Capell, both of Clayton State College

INTRODUCTORY ALGEBRA
Martha Wood and Peggy Capell, both of Clayton State College
James W. Hall, Parkland College

INTERMEDIATE ALGEBRA
Martha Wood and Peggy Capell, both of Clayton State College
James W. Hall, Parkland College

DEVELOPMENTAL MATHEMATICS, Third Edition
Martha Wood and Peggy Capell, both of Clayton State College
James W. Hall, Parkland College

GEOMETRY FOR COLLEGE STUDENTS
Peter B. Geltner and Darrell J. Peterson, both of Santa Monica College

PWS-KENT Publishing Company
20 Park Plaza, Boston, MA 02116
A Division of Wadsworth, Inc.
Partners in Education
1-800-343-2204 (in Mass. 617-542-3377)

WADSWORTH, INC. -- WINNER OF THE NATIONAL ASSOCIATION OF COLLEGE STORES
"Publisher of the Year"
Award for 1988!

MATHEMATICAL EXPOSITION

Shadow Prices and the Duality Principle of Linear Programming

by Sharon Kunoff
Long Island University/C.W. Post Campus
Brookville, NY 11548

Sharon received her Ph.D. in mathematics from Adelphi University in 1980. The title of her dissertation is "Asymptotic Analysis of Certain Ordinary Linear Differential Equations."

Linear programming is a standard topic in many finite mathematics courses for nonmathematics majors. It is often discussed following an introduction to matrices, along with a solution algorithm, the simplex method, which utilizes many of the techniques usually studied in the course. Alan Tucker presented a fine synopsis of the method in the wonderful interview with its founder, George Dantzig (Albers &Reid 1986). In the article, Dantzig indicated the continued relevance of the simplex method even with today's more sophisticated algorithms. Moreover, the simplex method should be the algorithm of choice in the kind of courses mentioned because it works very well for small problems and it is easily understood by the non-mathmajor student population.

In the elementary course, one usually sticks to problems in standard form, first solving the maximization problem using the simplex algorithm and then solving the minimization problem using the duality principle. The student is taught to set up the dual maximization problem associated with a given minimization problem by rote. Once this new problem is solved, using the simplex algorithm previously mastered, the student is told as in Wagner (1975) ". . . like the character in Moliere who is delighted to hear he has been speaking prose all his life without knowing it, you have already calculated dual solutions without knowing it . . . The coefficients of the slack variables in row 0 of the final simplex iteration of a maximizing problem are the optimum values of the dual variables." The advantage of solving minimization problems using the dual is that, in general, the hand calculations required of the students is considerably less than that required when solving standard minimization problems by other techniques. This advantage is paid for by the "mystery aspect" introduced. Usually no attempt is made to show why duality holds or why the solutions to the minimization problem appear where they do in the final simplex tableau. For example, Crown and Bittinger (1982) and Mizrachi and Sullivan (1973) simply say "do this", while Kenschaft (1973) and Wagner (1975) allude to shadow prices as justification. When the duality principle is proven by matrix methods as in Cooper and Steinberg (1974), the proof tends to be too sophisticated for the population addressed. It should be emphasized that the method presented in this article to set up and solve the dual problem is the usual one. The justification for presenting the shadow price argument to students is not to give them an alternate method for solving problems, but rather to give students with minimal mathematical background some of the true flavor of mathematics. That is, one attempts to convince them that mathematics is a discipline that emphasizes reasoning and proof, not rote manipulation of symbols. Thus, one might teach the dual

and then use the shadow price argument as a heuristic proof of the method, or one might teach shadow prices first and then show how this solves the dual problem as motivation for teaching the method.

In this article, shadow prices are defined in elementary terms. Using the definitions, a justification for the dualjty principle is presented which students find convincing. The heuristic argument presented explains why the simplex method solves both problems simultaneously and why the solutions to the minimization problem appear where they do in the standard simplex tableau. An offshoot of this approach is that it teaches students to formulate their own problems, which often helps them develop techniques for analyzing "verbal problems".

Shadow prices and shadow values will be defined in relation to actual problems. It will be seen that if the given problem is a maximization problem, the "shadow value" problem is the dual minimum problem. Conversely, if the original problem is a minimization, the "shadow price problem" is the dual maximum problem. Arguing from the definitions, it is shown that the maximum value of the function to be optimized in the original is the minimum value of the function to be minimized in the dual and vice versa.

Consider the following problem:

The A-1 Lighting Company manufactures three types of lamps: a high intensity lamp, a table lamp and a bed lamp. The profit on the high intensity lamps is $60 a dozen, the profit on the table lamps is $30 a dozen and the profit on the bed lamps is $20 a dozen. In the manufacturing process, a dozen high intensity lamps requires 1 hour on a wiring machine, 2 hours on an assembly line and 4 hours in a ceramic processor; a dozen table lamps requires 3 hours on the wirer, 3 hours on the assembly line and 2 hours in the processor; a dozen bed lamps requires 1 hour on the wirer, 3 hours on the assembly line and 1 hour in the processor. The wirer may only be in operation for 45 hours per week, the assembly line for 50 hours per week and the processor for 60 hours per week. Assuming that the company can sell all the lamps it produces, how many of each type lamp should the company produce each week for maximum profit?

Letting x_1, x_2 and x_3 represent, respectively, the number of dozen high intensity, table and bed lamps manufactured, the mathematical model of the problem becomes:

$$\begin{aligned} \text{Maximize profit } P = \quad & 60x_1 + 30x_2 + 20x_3 \\ \text{subject to :} \quad & x_1 + 3x_2 + x_3 \leq 45 \\ & 2x_1 + 3x_2 + 3x_3 \leq 50 \\ & 4x_1 + 2x_2 + x_3 \leq 60 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Next consider a problem associated with this one. Suppose the factory owner is considering allotting additional time for making lamps on one or more of the processor, the wirer or the assembly line. The owner would like to determine the worth or value of an additional hour's time on each of these processes when it is used in such a way as to optimally produce lamps. This value is known as the *marginal value* or *shadow value* associated with the process. In a sense, the time available on the given processes can be considered assets of the company, and the shadow values are the values associated with providing an additional hour of these assets when they are used in an optimal way for a specific purpose.

Let y_1 = the value of an additional hour of time on the wirer
y_2 = the value of an additional hour of time on the assembly line
y_3 = the value of an additional hour of time on the processor

when the wirer, assembly line and processor are used to produce lamps in the optimum manner, given the conditions of the maximization problem presented. The owner must determine the least it will cost to make additional time available to produce lamps. These costs (which we should emphasize are not real out of pocket costs) in the background, or shadows, are the shadow values associated with the assets: the wirer, assembly line and processor time. The owner, by not investing this time in some other productive venture, incurs a so-called "lost opportunity" cost and therefore wants to determine its worth, the minimum value of the assets when they are used optimally as indicated. Hence, we seek the minimum value of $C = 45y_1 + 50y_2 + 60y_3$, the sum of the products of the time available for the process by the worth per hour of the process, if additional time were to be made available.

Note that it is the time made available to produce lamps that yields the profit on the lamps. Hence, the value of the total time spent in manufacturing a dozen of each type of lamp must be worth at least as much as the profit on those lamps. In particular, since one dozen high intensity lamps requires 1 hour on the wirer, 2 hours on the assembly line and 4 hours in the ceramic processor and yields a $60 profit, we must have $y_1 + 2y_2 + 4y_3 \geq 60$; that is, the sum of the products of the hours spent in the process with the worth of an hour's time on the process yields the value of the time spent in manufacturing high intensity lamps. This value must be worth at least as much as a dozen of the lamps. The corresponding expressions for the table lamps and bed lamps respectively are $3y_1 + 3y_2 + 2y_3 \geq 30$ and $y_1 + 3y_2 + y_3 \geq 20$. Of course, all the "worths" are nonnegative, yielding a new linear programming problem:

$$
\begin{array}{lrcrcrcl}
\text{Minimize } C = & 45y_1 & + & 50y_2 & + & 60y_3 & & \\
\text{subject to :} & y_1 & + & 2y_2 & + & 4y_3 & \geq & 60 \\
& 3y_1 & + & 3y_2 & + & 2y_3 & \geq & 30 \\
& y_1 & + & 3y_2 & + & y_3 & \geq & 20
\end{array}
$$

$$y_1, y_2, y_3 \geq 0$$

In matrix form, the following shows the original problem and the shadow value problem side by side:

Original

$$\text{Maximize } P = \begin{bmatrix} 60 & 30 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

subject to:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 3 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 45 \\ 50 \\ 60 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq 0$$

Shadow Value

$$\text{Minimize } C = \begin{bmatrix} 45 & 50 & 60 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

subject to:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 60 \\ 30 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq 0$$

The shadow value problem is the dual of the original problem in the standard mathematical sense, namely, if the primal problem is the following in matrix notation:

$$\begin{aligned} &\text{Maximize} && P = \mathbf{RX} \\ &\text{subject to:} && \mathbf{AX} \leq \mathbf{B}\ ,\ \mathbf{X} \geq 0 \end{aligned}$$

Then, using $\mathbf{A}^T$ to stand for A-transpose, the dual problem is:

$$\begin{aligned} &\text{Minimize} && C = \mathbf{B}^T\mathbf{Y} \\ &\text{subject to:} && \mathbf{A}^T\mathbf{Y} \geq \mathbf{R}^T\ ,\ \mathbf{Y} \geq 0 \end{aligned}$$

The theorem which will be justified by the shadow value argument states: if **X** and **Y** are each feasible solutions to their respective problems and if $\mathbf{RX} = \mathbf{B}^T\mathbf{Y}$, then both **X** and **Y** are optimal solutions to their respective problems and Max P = Min C.

The Shadow Value Argument

The original problem was that of determining the maximum profit that a corporation can obtain from its assets; in this case the time available on a wirer, assembly line and a ceramic processor, if the time is used to produce the lamps specified in an optimal manner. The minimum value C, the value of the total time available, cannot be worth less than the profit it generates, when the lamps are produced in the optimum manner, so Min $C \geq$ Max P. Conversely, if the profits aren't worth at least as much as the assets dedicated to producing these profits, the assets aren't being used effectively. This implies Max $P \geq$ Min C. The only way both statements can be true is if Max P = Min C.

If the original problem is solved by the simplex method in the manner described by Tucker (Albers and Reid, 1986), the initial tableau for the problem can be written:

x_1	x_2	x_3	s_1	s_2	s_3	
1	3	1	1	0	0	45
2	3	3	0	1	0	50
4	2	1	0	0	1	60
-60	-30	-20	0	0	0	-P

Notice that the slack variable columns have been labeled s_1, s_2 and s_3 to distinguish them from the actual variable columns and that the objective variable row has a -P in the last column so that it translates directly to an equation in the same manner as the remaining rows in the matrix. After applying the simplex method, the final tableau for the problem becomes:

x_1	x_2	x_3	s_1	s_2	s_3	
0	1.9	0	1	-.3	-.1	24
0	.8	1	0	.4	.2	8
1	.3	0	0	-.1	.3	13
0	4.0	0	0	2.0	14.0	-P + 940

The equation represented by the final row of the tableau reads: $4x_2 + 2s_2 + 14s_3 = -P + 940$, or $P = 940 - 4x_2 - 2s_2 - 14s_3$. This implies that the largest value P can attain is 940

when x_2, s_2 and s_3 are all zero. Setting these variables to zero and reading the equations now represented by the tableau row by row yields: $s_1 = 24$, $x_3 = 8$ and $x_1 = 13$. This is the solution to the original (maximization) problem; manufacture 13 dozen high intensity lamps and 8 dozen bed lamps for a maximum profit of $940. The fact that $s_1 = 24$ implies that there are 24 hours of unused wirer time available when the processes are used to produce lamps in the optimum manner since the slack variables s_1, s_2 and s_3 are associated with the time available on the wirer, assembly line and ceramic processor, respectively.

As has already been shown, the solution to the shadow value problem (the dual minimum problem) must have Minimum C = Maximum P, so that the available time is worth $940. Since when P is a maximum there are 24 hours still available on the wirer, using additional time adds nothing to the profit so *additional* wirer time is worth nothing, or $y_1 = 0$. This is signified in the tableau by the fact that the coefficient of s_1 in the final row (the number under s_1) is zero. Since the final row of the tableau led to the equation $P = 940 - 4x_2 - 2s_2 - 14s_3$, if $s_2 \neq 0$, for each hour the assembly line is not used P decreases by $2 so that time on this assembly line is worth $2 per hour when it is used to produce lamps in the optimum manner. Thus, $y_2 = 2$ and this value is the number under the s_2 column of the final row. Similarly, if $s_3 \neq 0$, since the coefficient of s_3 is 14, P decreases by $14 for each unused hour in the processor, so that processor time is worth $14 when used to produce lamps in the optimum manner. Thus $y_3 = 14$ and this number appears beneath the s_3 column in the final tableau. In summary, the final simplex tableau not only contains the solution to the original maximization problem, but it also contains the solution to the dual minimization (shadow value) problem. Moreover, it is not at all surprising that the solutions occur in the positions noted in the final tableau. When setting up the tableau, it may be helpful to students to actually write y_i above s_i to remind them of the relationship between the two sets of variables and where to find the required values in the tableau.

Minimization as the Primal Problem

Consider the following problem:

HEP Health Foods Corporation is planning to market a granola product made from three principle ingredients: oats, sweetened powdered milk and raisins. The company plans to advertise that its product contains at least 6 units of nutrient A and 7 units of nutrient B per serving. It pays 9 cents an ounce for oats, 12 cents an ounce for the milk product and 26 cents an ounce for dried raisins. An ounce of oats contains 1 unit of nutrient A and 2 units of nutrient B, an ounce of powdered milk contains 3 units of A and 1 unit of B, and an ounce of raisins contains 4 units of A and 3 units of B. How many ounces of each ingredient should HEP put into a serving of its product to meet the advertised claims at the least cost?

For consistency in notation, if we let y_1, y_2 and y_3 stand for the number of ounces of oats, milk and raisins respectively per serving of the product, the mathematical model of the problem becomes:

$$\text{Minimize cost (in cents) } C = 9y_1 + 12y_2 + 26y_3$$

$$\text{subject to : } \quad y_1 + 3y_2 + 4y_3 \geq 6$$

$$2y_1 + y_2 + 3y_3 \geq 7$$

$$y_1, y_2, y_3 \geq 0$$

A shadow price problem can be developed with respect to this minimization problem. Essentially, HEP must meet advertised claims for nutrients, so they can be said to be selling nutrients. They are interested in the worth of these nutrients when they are obtained from the given ingredients in the optimum manner. One might consider the possibility that HEP sell pure nutrients since they are what makes the product saleable. The prices that the nutrients would sell for, the values of the nutrients lurking in the shadows, so to speak, are the shadow prices associated with the nutrients. HEP would want to obtain the maximum income possible from the sale of the nutrients if they were to be sold. If x_1 = the worth in cents of a unit of nutrient A and x_2 = the worth in cents of a unit of nutrient B when they are derived from oats, powdered milk and raisins purchased in the optimum manner consistent with the given problem, then the goal is to find the maximum value of the 6 units of nutrient A and 7 units of nutrient B under these conditions. That is, maximize $6x_1 + 7x_2$. Since the nutrients are obtained from specified ingredients, the worth of the nutrients in a particular food cannot exceed the value of the food. In particular, oats contain 1 unit of nutrient A and 2 units of nutrient B, so $x_1 + 2x_2 \leq 9$; that is the sum of the products of the number of units of the nutrient in the oats with the value of the nutrient cannot exceed the cost of the oats. Similarly, for powdered milk $3x_1 + x_2 \leq 12$ and for raisins $4x_1 + 3x_2 \leq 26$. Again all the "worths" are nonnegative, so that in this case the shadow price problem becomes:

$$
\begin{array}{lrcrcl}
\text{Maximize } P = & 6x_1 & + & 7x_2 & & \\
\text{subject to :} & x_1 & + & 2x_2 & \leq & 9 \\
 & 3x_1 & + & x_2 & \leq & 12 \\
 & 4x_1 & + & 3x_2 & \leq & 26
\end{array}
$$

$$x_1, x_2 \geq 0$$

This maximization problem is the dual of the given minimization (primal) problem. Since the usual simplex method solves maximization problems, for this problem the simplex method is set up on the dual. The slack variables s_1, s_2 and s_3 refer to the value of the oats, milk and raisins, respectively. The initial simplex tableau becomes:

		y_1	y_2	y_3	
x_1	x_2	s_1	s_2	s_3	
1	2	1	0	0	9
3	1	0	1	0	12
4	3	0	0	1	26
-6	-7	0	0	0	-P

After several steps the final tableau becomes:

		y_1	y_2	y_3	
x_1	x_2	s_1	s_2	s_3	
0	1	.6	-.2	0	3
1	0	-.2	.4	0	3
0	0	-1.0	-1.0	1	5
0	0	3.0	1.0	0	-P + 39

Although in this case the minimization problem was the primal problem, it is also the dual of the shadow price problem. In that sense, mathematically it can be thought of in the same way as the dual minimum of the previous example. Thus, the amounts of oats, milk and raisins to be used in a serving appear under the slack variables associated with these ingredients. Reading the numbers under s_1, s_2 and s_2 respectively yields the information that HEP should use 3 ounces of oats, 1 ounce of powdered milk and no raisins for a minimum cost of 39 cents per serving.

Here the shadow price problem was solved directly. If s_1 and s_2 are both zero, P is a maximum and the rows of the tableau yield: $x_2 = 3$, $x_1 = 3$ and $s_3 = 5$. Interpreted in terms of shadow prices, the maximum income to be derived from the sale of nutrients A and B if they are obtained from the given ingredients in the optimum manner is 39 cents and they can be sold for 3 cents a unit each.

Summary

Associated with each primal linear programming problem in standard form there is a dual problem. If the primal problem is a maximization problem, the dual problem is a minimization problem and vice versa. Shadow prices or shadow values are the values of the variables in the dual problem. In terms of actual problems, these values often have economic significance. If the primal problem is a maximization problem, the shadow values usually represent the worth of the assets of the company in terms of the use to which they are put. If the primal problem is a minimization problem, the shadow prices can often represent the selling price of the basic products (perhaps the "shadow products") obtained from the materials purchased. Encouraging students to understand the shadow price argument helps them to understand why, when the standard simplex tableau is set up for the maximization problem and is solved in the usual manner, the optimum value is the same for the maximization and the associated minimization problems, and why the solutions for the variables of the minimization problem appear under the slack variables in the final simplex tableau.

References

Albers, D. J., & Reid, C. (1986). An interview with George B. Dantzig the father of linear programming. *The College Mathematics Journal, 17*(4), 292-314.

Crown, J. C., & Bittinger, M. (1982). *Finite mathematics, a modeling approach* (2nd ed.). Reading, MA: Addison-Wesley.

Cooper, L., & Steinberg, D. (1974). *Methods and applications of linear programming*. Philadelphia, PA: Saunders.

Kenschaft, P. C. (1973). *Linear mathematics a practical approach*. New York: Worth.

Mizrachi, A., & Sullivan, M. (1973). *Mathematics for business and social science, an applied approach* (2nd ed.). New York: Wiley.

Wagner, H. M. (1975). *Principles of operations research* (2nd ed.). Englewood Cliffs, NJ: Prentice-Hall.

Put Our Books

DEVELOPMENTAL MATHEMATICS

INTRODUCTORY ALGEBRA, *2/e, 1987*
INTERMEDIATE ALGEBRA, *2/e, 1987*
both by Linda R. Pulsinelli/Patricia I. Hooper

PRECALCULUS MATHEMATICS

COLLEGE ALGEBRA, *1987*
COLLEGE ALGEBRA AND TRIGONOMETRY, *1987*
TRIGONOMETRY, *1988*
PRECALCULUS, *1988*
all by Michael Sullivan

ESSENTIALS OF TRIGONOMETRY, *4/e, 1986*
TRIGONOMETRY: An Analytic Approach, *5/e, 1987*
both by Irving Drooyan/Walter Hadel/Charles C. Carico

LIBERAL ARTS MATHEMATICS

FUNDAMENTALS OF MATHEMATICS, *5/e, 1989*
William M. Setek

TOPICS IN CONTEMPORARY MATHEMATICS, *4/e, 1989*
Jack Britton/Ignacio Bello

MATHEMATICS FOR ELEMENTARY TEACHERS

MATHEMATICS FOR ELEMENTARY TEACHERS, *1988*
Gary L. Musser/William F. Burger

APPLIED MATHEMATICS

FINITE MATHEMATICS FOR MANAGEMENT, LIFE AND SOCIAL SCIENCES, *4/e, 1987*

APPLIED MATHEMATICS FOR MANAGEMENT, LIFE AND SOCIAL SCIENCES, *3/e, 1989*

ESSENTIAL COLLEGE MATHEMATICS FOR BUSINESS AND ECONOMICS, LIFE SCIENCES AND SOCIAL SCIENCE, *1989*

COLLEGE MATHEMATICS FOR MANAGEMENT, LIFE AND SOCIAL SCIENCES, *4/e, 1987*

CALCULUS FOR MANAGEMENT, LIFE AND SOCIAL SCIENCES, *4/e, 1987*

APPLIED CALCULUS FOR MANAGEMENT, LIFE AND SOCIAL SCIENCES, *3/e, 1988*
all by Raymond A. Barnett/Michael R. Ziegler

MACMILLAN/DELLEN

to Work for You

DISCRETE MATHEMATICS

ESSENTIAL DISCRETE MATHEMATICS, *1987*
Richard Johnsonbaugh

DISCRETE AND COMBINATORIAL MATHEMATICS, *1987*
Abraham Hillman/Gerald L. Alexanderson/Richard Grassl

CALCULUS AND LINEAR ALGEBRA

CALCULUS AND ANALYTIC GEOMETRY, *1988*
Lynn E. Garner

INTRODUCTORY LINEAR ALGEBRA WITH APPLICATIONS, *4/e, 1988*
Bernard Kolman

A UNIFIED INTRODUCTION TO LINEAR ALGEBRA, *1988*
Alan C. Tucker

LINEAR ALGEBRA, *1987*
Raymond A. Barnett/Michael R. Ziegler

STATISTICS

STATISTICS, *4/e, 1988*
A FIRST COURSE IN STATISTICS, *3/e, 1989*
both by James T. McClave/Frank H. Dietrich II

STATISTICS BY EXAMPLE, *3/e, 1987*
Terry Sincich

BASIC STATISTICS: An Inferential Approach, *3/e, 1989*
Frank H. Dietrich II/Thomas J. Kearns

COMPUTER SCIENCE

PASCAL: Programming and Problem Solving, *2/e, 1987*
Sanford Leestma/Larry Nyhoff

PROBLEM SOLVING WITH PASCAL, *1988*
Hugo T. D'Alarcao/Robert Sutherland

ASSEMBLY LANGUAGE FOR THE IBM-PC, *1989*
Kip R. Irvine

QUICK BASIC: Programming and Its Applications for the IBM-PC, *1989*
David Schneider

APPLICATIONS PROGRAMMING IN C, *1989*
Richard Johnsonbaugh/Martin Kalin

SEE HOW OUR BOOKS WORK FOR YOU:

VISIT BOOTH T-3 AT THE AMATYC MEETING,
CONTACT YOUR LOCAL SALES REPRESENTATIVE, OR CALL (800) 428-3750.

MACMILLAN PUBLISHING COMPANY • COLLEGE DIVISION/866 THIRD AVE., NEW YORK, NY 10022

DELLEN/MACMILLAN

Double-Angle Triangles, Pentagons, Paradoxes, and Gold

by
Bill Leonard
California State University
Fullerton, CA 92634

Jan Schwartzman
University High School
Fullerton, CA 92634

Harris Shultz
California State University
Fullerton, CA 92634

Bill Leonard is the author of over 25 published articles and two books and has been an invited speaker at mathematics conferences and symposia more than 200 times, including keynote speaker for more than a dozen different organizations. Jan Schwartzman, former Mathematics Department Chair at University High School (Irvine, California), serves on the instructional staff of the University of California, Irvine, Summer Mathematics Institute and is a visiting lecturer at California State University, Fullerton. Harris Shultz, author of over 35 published articles, has directed three National Science Foundation summer institutes for pre-college teacher enhancement and has been an invited participant at several international conferences.

The triples 3-4-5 and 5-12-13 are familiar to us as the sides of right triangles. Considerably less familiar to most of us is the 4-5-6 triangle. The reader might wonder what is interesting about this non-right, non-equilateral, non-isosceles triangle. Not at all obvious nor well-known is the fact that its largest angle is exactly twice its smallest angle. In this article we shall show that such is the case, investigate other triangles having a similar property, and see how the golden ratio and its relation to pentagons and geometric paradoxes is a natural outcome of such an investigation.

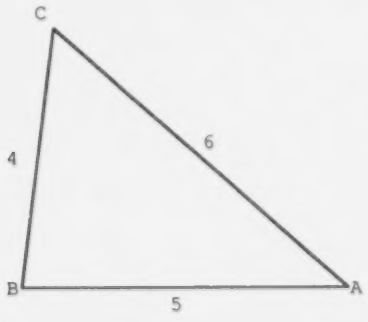


Figure 1

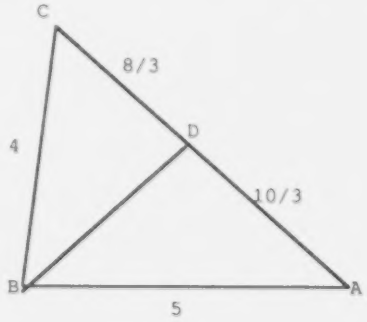


Figure 2

In Figure 1 we label a triangle ABC as shown. In Figure 2 the angle B is bisected. Since an angle bisector of a triangle divides the opposite side into segments whose lengths form the same ratio as the adjacent sides (4 to 5 in this case), segments CD and DA must have lengths 8/3 and 10/3 respectively. Now, the proportion

$$\frac{4}{\frac{8}{3}} = \frac{6}{4}$$

tells us that triangle BCD is similar to triangle ACB. Hence, angle CBD is congruent to angle CAB and, therefore, the measure of angle CBA is twice that of angle CAB. That is, in a 4-5-6 triangle, the angle opposite the side of length 6 is twice the angle opposite the side of length 4.

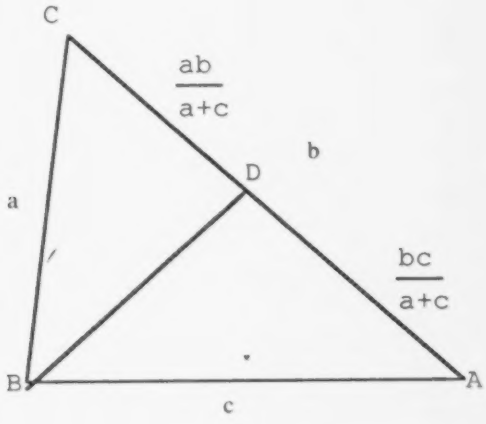


Figure 3

We now ask the question, given a triangle ABC with corresponding sides a, b, and c, under what conditions can we conclude that angle B is twice angle A? Looking at Figure 3 and using the motivation of the 4-5-6 triangle, we can see that triangle BCD is similar to triangle ACB is the proportion

$$\frac{a}{\frac{ab}{a+c}} = \frac{b}{a}$$

is true. In such a case, angle CBD will be congruent to angle CAB and, therefore, the measure of CBA is twice that of CAB. Now, the above proportion is equivalent to the equation

$$b^2 = a(a+c).$$

In addition to this equation relating a, b, and c, it is necessary that $a < b + c$, $b < a + c$, and $c < a + b$. The first of these is assured since $a < b$. But $a < b$ also tells us that

$$ab < b^2 = a(a+c)$$

and, therefore, $b < a + c$. The condition that $c < a + b$ is equivalent to $c - a < b$. Squaring both sides yields

$$c^2 - 2ac + a^2 < b^2 = a^2 + ac,$$

which simplifies to $c < 3a$. This argument is easily reversible. That is, if $b^2 = a(a+c)$ and $c < 3a$, then

$$(c-a)^2 = c^2 - 3ac + a^2 + ac < a^2 + ac = b^2,$$

so that $|c-a| < b$ and, therefore, $c < a + b$. Thus angle B is twice angle A if and only if

$$b^2 = a(a+c) \text{ and } c < 3a.$$

If a, b, and c are natural numbers having no common divisors other than 1, then a and $a+c$ must be relatively prime squares. Table 1 gives several triangles ABC having the property that angle B is twice angle A.

a	a + c	b^2	b	c
4	9	36	6	5
9	16	144	12	7
9	25	225	15	16
16	25	400	20	9

Table 1

A special case is an isosceles triangle in which angle A is 36 degrees and angles B and C are 72 degrees (Figure 4). Since $b = c$, our equation becomes

$$b^2 = a(a+b).$$

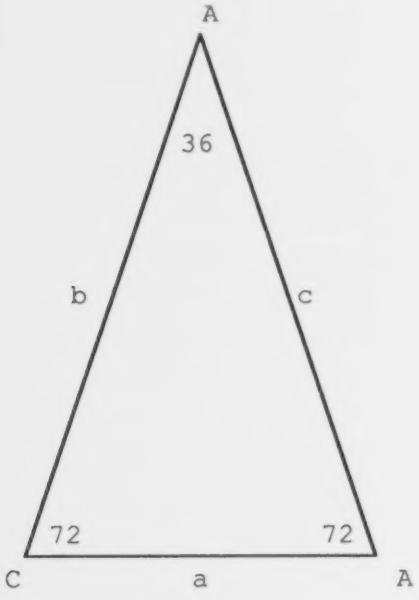


Figure 4

31

Solving for a / b using the quadratic formula, we find that

$$\frac{a}{b} = \frac{\sqrt{5} - 1}{2},$$

the golden ratio. Thus, in an isosceles triangle having base angles 72, the ratio of the base to either of the equal sides is the golden ratio.

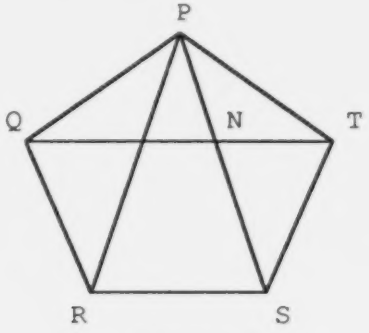


Figure 5

Consider now a regular pentagon PQRST. In Figure 5 we draw diagonals PR and PS and notice that angle RPS is an inscribed angle subtending 1 / 5 of the circumscribed circle and, therefore, has measure 36 degrees. Thus, triangle RPS is an isosceles triangle having 72 degree base angles. It follows immediately from the above discussion that RS to PR, that is, the ratio of a side of a regular pentagon to a diagonal, is the golden ratio. But there is more. Since NQRS is a parallelogram, QN = RS and, therefore,

$$\frac{QN}{QT} = \text{golden ratio.}$$

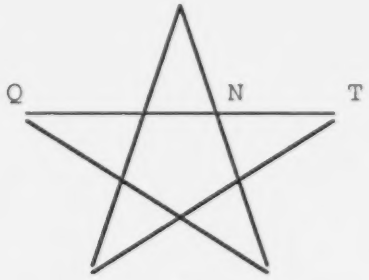


Figure 6

Thus, in the regular five-pointed star in Figure 6, we have

$$\frac{QN}{QT} = \text{golden ratio.}$$

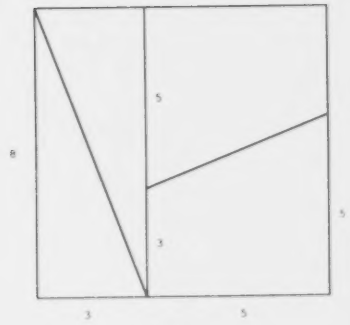


Figure 7

Figure 8

In our final investigation, the four pieces that form the square in Figure 7 are rearranged as shown in Figure 8. There is an apparent paradox in that the square has area 64 while the rectangle appears to have area 65. This contradiction of the principle of conservation of area is resolved when one notices that, contrary to what is suggested by Figure 8, the four pieces do not quite fit together to form a rectangle. In Figure 9 we try to generalize this apparent paradox. If a and b are chosen to be consecutive terms of the Fibonacci sequence 1,1,2,3,5,8, . . ., then $b - a$ immediately precedes a and $a + b$ immediately follows b in that sequence; moreover, it is a well known property of the Fibonacci sequence that $a(a + b) - b^2 = \pm 1$. Thus, the "rectangle" in Figure 10 will have area that is 1 less or 1 greater than the area of the square in Figure 9. One might ask if the converse is true. That is, if the "rectangle" in Figure 10 has area that is 1 less or 1 greater than the area of the square in Figure 9, does it necessarily follow that a and b are consecutive terms of the Fibonacci sequence? The answer is yes and can be proven by induction as follows.

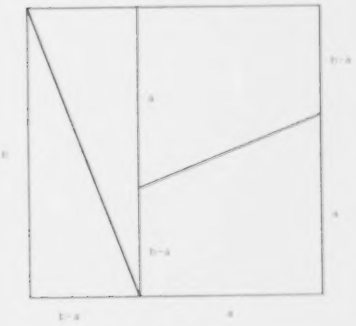


Figure 9

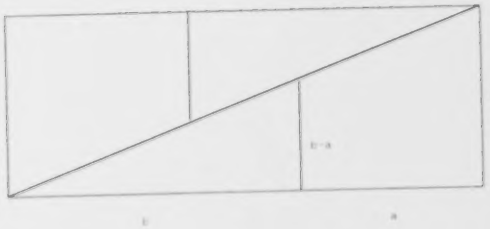


Figure 10

Proof: Suppose that $a = 1$ and we have $a(a + b) - b^2 = \pm 1$. Then, $1 + b - b^2 = \pm 1$, so that either $b = b^2$ or $b^2 - b - 2 = 0$. Since $b > 1$, the only possibility is $b = 2$. Thus, the assertion that a and b must be consecutive terms of the Fibonacci sequence is true if $a = 1$. Suppose now that the assertion is true for all $a \leq k$. If $a(a + b) - b^2 = \pm 1$ where $a = k + 1$, then since $b > a > 1$, we have

$$b^2 - 2ab + 1 < b^2 - 2ab + a(b-a) = b^2 - ab - a^2 \leq 1,$$

and, therefore, $b^2 < 2ab$. It then follows that $b - a < a$. That is, $b - a \leq k$. Moreover,

$(b-a)[(b-a)+a]-a^2 = b^2 - ab - a^2 = \pm 1$. By the induction hypothesis, $b-a$ and a must be consecutive terms of the Fibonacci sequence. Since $b = a + (b-a)$, the same must be true of a and b.

Conclusion

We conclude by asking under what conditions will the four pieces in Figure 10 actually fit together to form a rectangle (having, of course, the same area as in Figure 9). If such is to be the case, all the right triangles in Figure 10 must be similar. This will occur if and only if

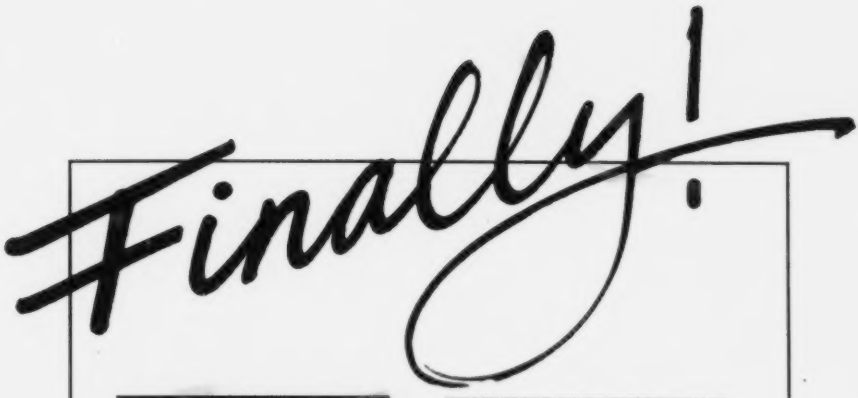
$$\frac{a}{a+b} = \frac{b-a}{b}.$$

This is equivalent to $a^2 + ab - b^2 = 0$, from which we conclude that a/b must be the ubiquitous golden ratio.

IT'S COMING!

1988 AMATYC CONVENTION

Calgary
Alberta, Canada
Oct. 27th — Oct. 30th, 1988



MATH TESTING

MADE SIMPLE
MADE QUICK
MADE TO ORDER

IPSTEST

Algorithm-driven software based on YOUR OBJECTIVES

- District or state competency requirements
- Texbook objectives
- Departmental standards
- Personal selection

For further information and a FREE DEMONSTRATION DISK

contact:

ips Publishing, Inc.
31316 Via Colinas, Suite 110
Westlake Village, CA 91362
(818) 706-1646

Please specify Apple or IBM

Some Applications of *Almost Geometric* Infinite Series

by
Coreen Mett and Susan Milton
Radford University, Radford, Virginia 24142

Dr. Coreen Mett received her B.S. degree in physics and mathematics from Mankato State University in Minnesota and her Ph.D. in mathematics from Michigan State University. She has fallen in love with the mountains of western Virginia, and has taught both at Virginia Polytechnic Institute and State University and at Radford University for the past 15 years. She is a co-author of an applied calculus text. Her free time is spent hiking Appalachian trails, gardening, and enjoying a variety of sports. Dr. J. S. Milton holds the M.A. degree in mathematics from UNC-Chapel Hill and the Ph. D. in statistics from Virginia Polytechnic Institute and State University. She has taught undergraduate mathematics and statistics courses for twenty years and is the author of statistics texts ranging from general to graduate level. She is currently serving as Professor of statistics and co-ordinator of the statistical consulting service at Radford University, Radford, Virginia.

Introduction

In beginning mathematics courses it is sometimes difficult to provide realistic applications of the techniques being studied. However, the importance of infinite series can be underscored by demonstrating that they occur quite naturally in physical settings. In this article an interesting method for evaluating a series which arises in the context of discrete probability is developed. The series is called an *almost geometric* series. The method used in the probability example is then generalized to apply to all *almost geometric* series. Other examples are presented as exercises for the reader.

An *Almost Geometric* Series in Probability

Consider a gambler who intends to play roulette, betting on red until he wins at which time he will quit. Associated with this experiment is a variable X, the number of losses that occur before the first win. This variable is "random" in the sense that its value is determined by chance. A *priori*, before the fact, X can assume any of the values 0, 1, 2, 3, ... It is easy to derive a function, called the probability density, that gives the probability that X will assume each of these values. To do so, we need only note that a roulette wheel consists of thirty-eight slots, each of the same size. Eighteen of these slots are red, eighteen are black, and two are green. When the wheel is spun a ball comes to rest in one of these slots, thereby determining the winning number. Let *s* denote *success*, obtaining a red number, and let *f* denote *failure*, obtaining a number that is not red. The possible outcomes for this experiment together with the resulting values of X are given below:

Outcome:	*s*	*fs*	*ffs*	*fffs*	*ffffs*	...
	↓	↓	↓	↓	↓	
Value for X:	0	1	2	3	4	...

Since 18 of the 38 slots are red, the probability of success on each spin is 18/38 and the probability of failure is 20/38. Since the result of one spin has no effect on the subsequent results, the outcomes are independent. Consequently the probability of an event such as *ffs* is obtained by multiplying probabilities of individual outcomes

$$\frac{20}{30} \cdot \frac{20}{38} \cdot \frac{18}{38} .$$

Some of these probabilities are listed in the following table:

x	0	1	2	3	...
$P[X = x]$	$\frac{18}{38}$	$\frac{20}{38} \cdot \frac{18}{38}$	$\frac{20}{38} \cdot \frac{20}{38} \cdot \frac{18}{38}$	$\frac{20}{38} \cdot \frac{20}{38} \cdot \frac{20}{38} \cdot \frac{18}{38}$	...

There is a definite pattern to the probabilities listed in the second row of this table. This pattern can be formalized to define a function, f, called the density for X. In particular, f is defined by

$$f(x) = \left(\frac{20}{38}\right)^x \frac{18}{38} \; ; \; x = 0,1,2,3,\ldots .$$

Note that X must assume one of the values $x = 0,1,2,3,\ldots$. Thus, if we sum the probabilities associated with these possible values we must get 1. That is, it is necessary that

$$\sum_{x=0}^{\infty} \left(\frac{20}{38}\right)^x \frac{18}{38} = 1 .$$

To verify that this is true, we note that the above series is a geometric series with first term 18/38 and common ratio 20/38. Thus its sum is

$$\frac{\frac{18}{38}}{1 - \frac{20}{38}} = 1 .$$

The next question to ask is, what is the theoretical average number of losses that occur before the first win? That is, what is the expected value for X? The expected value, denoted by E[X], is defined by

$$E[X] = \sum_{x=0}^{\infty} x \cdot f(x) = \sum_{x=0}^{\infty} x \cdot \left(\frac{20}{38}\right)^x \frac{18}{38} .$$

Expanding this series, we see that

$$E[X] = 0 + 1 \cdot \left(\frac{20}{38}\right)\frac{18}{38} + 2 \cdot \left(\frac{20}{38}\right)^2 \frac{18}{38} + 3 \cdot \left(\frac{20}{38}\right)^3 \frac{18}{38} + 4 \cdot \left(\frac{20}{38}\right)^4 \frac{18}{38} + \ldots . \qquad (1)$$

This series is *not* geometric and cannot be evaluated without some thought. To evaluate the series, consider the series

$$\left(\frac{20}{38}\right) E[X] = \left(\frac{20}{38}\right)^2 \frac{18}{38} + 2 \cdot \left(\frac{20}{38}\right)^3 \frac{18}{38} + 3 \cdot \left(\frac{20}{38}\right)^4 \frac{18}{38} + \ldots . \qquad (2)$$

We now subtract line (2) from line (1) to obtain

$$\left(\frac{18}{38}\right) E[X] = \left(\frac{20}{38}\right)\frac{18}{38} + \left(\frac{20}{38}\right)^2 \frac{18}{38} + \left(\frac{20}{38}\right)^3 \frac{18}{38} + \left(\frac{20}{38}\right)^4 \frac{18}{38} + \ldots .$$

The series on the right is geometric with first term $(\frac{20}{38})\frac{18}{38}$ and common ratio $\frac{20}{38}$ and sums to

$$\frac{\left(\frac{20}{38}\right)\frac{18}{38}}{1-\frac{20}{38}} = \frac{20}{38}.$$

We thus obtain the equation

$$\frac{18}{38}E[X] = \frac{20}{38},$$

or E[X] = 20/18. That is, if the gambler used this betting strategy over and over again, the average number of failures that would occur in the long run is 20/18 or a little more than one. Since betting on red in roulette is an even money bet, this strategy, in the long run, will result in a loss to the gambler!

A Generalization

Based on our introductory example, we now define an *almost geometric* series to be any series of the form

$$\sum_{x=1}^{\infty} xr^x .$$

Our method for evaluating an *almost geometric* series is generalized as follows: Consider any series of the form

$$S = \sum_{x=1}^{\infty} xr^x , \text{ where } |r| < 1 . \tag{3}$$

Consider also the series

$$rS = r\sum_{x=1}^{\infty} xr^x . \tag{4}$$

The latter can be rewritten

$$r\sum_{x=1}^{\infty} xr^x = \sum_{x=1}^{\infty} xr^{x+1} = \sum_{x=2}^{\infty} (x-1)r^x = \sum_{x=1}^{\infty} (x-1)\,r^x .$$

Then subtracting (4) from (3) we obtain

$$S - rS = \sum_{x=1}^{\infty} xr^x - \sum_{x=1}^{\infty} (x-1)r^x = \sum_{x=1}^{\infty} [x-(x-1)]r^x = \sum_{x=1}^{\infty} r^x .$$

Since the resulting series is geometric, with $|r| < 1$,

$$S - rS = \frac{r}{1-r}$$

and hence

$$S = \frac{r}{(1-r)^2} .$$

Therefore, the sum of an *almost geometric* series is

$$S = \sum_{x=1}^{\infty} xr^x = \frac{r}{(1-r)^2} \quad \text{when } |r| > 1 . \tag{5}$$

In our gambler's example, we evaluated the series

$$\sum_{x=0}^{\infty} x \left(\frac{20}{38}\right)^x \frac{18}{38}.$$

This series can be rewritten as

$$\sum_{x=0}^{\infty} x \left(\frac{20}{38}\right)^x \frac{18}{38} = \left(\frac{18}{38}\right) \sum_{x=1}^{\infty} x \left(\frac{20}{38}\right)^x .$$

Applying the results of equation (5) with r = 20/38, we see that

$$E[X] = \frac{18}{38} \frac{\frac{20}{38}}{(1-\frac{20}{38})^2} = \frac{20}{38}.$$

This result, of course, agrees with that obtained previously.

Other Applications

Although this procedure first came to our attention in the context shown, namely that of trying to find E[X] for a discrete random variable, other applications exist. We state two of these without proof, leaving the verification as an exercise for the reader.

Application 1. Let X be a random variable with density

$$f(x) = c(r)^x ; \quad x = 1,2,3,\ldots ; \quad 0 < r < 1 .$$

The n*th* moment of X about the origin is given by

$$E[X^n] = \sum_{x=1}^{\infty} x^n cr^x .$$

This series is a generalization of our *almost geometric* series. A recursive formula for $E[X^n]$ can be derived by considering $E[X^n] - rE[X^n]$. This formula is given by

$$E[X^n] = \frac{\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} E[X^{n-k}]}{1-r} , \text{ where } \binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ and } E[1] = 1 . \tag{6}$$

For example, $E[X] = \frac{1}{1-r}$, $E[X^2] = \frac{1+r}{(1-r)^2}$, $E[X^3] = \frac{1+4r+r^2}{(1-r)^3}$.

Application 2. The n^{th} moment $E[X^n] = \sum_{x=1}^{\infty} x^n cr^x$ is analogous to the power series $S_n(x) = \sum_{k=1}^{\infty} k^n x^k$. The power series $S_n(x)$ clearly converges for $|x| < 1$. In the special case n = 0, the formula for a geometric series gives the closed form

$$S_0(x) = \sum_{k=1}^{\infty} x^k = \frac{x}{1-x}, \quad |x| < 1 .$$

Our method of examining $S_n(x) - xS_n(x)$, or an alternative method using the derivative relationship

$$S_{n+1}(x) = xS'_n(x), \quad |x| < 1 ,$$

generates the closed forms

$$S_1(x) = \frac{x}{(1-x)^2}$$

$$S_2(x) = \frac{x(1+x)}{(1-x)^3}$$

$$S_3(x) = \frac{x(1+4x+x^2)}{(1-x)^4}$$

$$S_4(x) = \frac{x(1+11x+11x^2+x^3)}{(1-x)^5}$$

$$\vdots$$

Notice that for $n \geq 1$

$$S_n(x) = \frac{xQ_n(x)}{(1-x)^{n+1}} ,$$

where $Q_n(x)$ is a polynomial of degree n-1.

$$Q_1(x) = 1$$
$$Q_2(x) = 1 + x$$
$$Q_3(x) = 1 + 4x + x^2$$
$$Q_4(x) = 1 + 11x + 11x^2 + x^3$$
$$\vdots \quad .$$

In a recent paper, Segal (1983) observed that the coefficients in $Q_n(x)$ form a pattern similar to the binomial coefficients in Pascal's Triangle. That is, the coefficients of $Q_{n+1}(x)$ can be obtained from the coefficients of $Q_n(x)$.

```
                    1
                1       1
            1       4       1
        1       11      11      1
    1       26      66      26      1
:       :       :       :       :       :
```

Let $Q_n(x) = \sum_{k=0}^{n-1} c_k x^k$ and $Q_{n+1}(x) = \sum_{k=0}^{n} d_k x^k$.

Then the derivative relationship

$$S_{n+1}(x) = xS'_n(x)\,,$$

can be used to obtain the following pattern for coefficients:

$$\begin{aligned} d_0 &= c_0 \\ d_k &= (1+k)c_k + (n-k+1)c_{k-1}\,; \quad k = 1,2,3,\ldots,n-1 \\ d_n &= c_{n-1}. \end{aligned}$$

For example, when $n = 3$ the polynomial $Q_3(x) = 1 + 4x + x^2$ generates the polynomial $Q_4(x) = d_0 + d_1x + d_2x^2 + d_3x^3$, where

$$\begin{aligned} d_0 &= c_0 = 1 \\ d_1 &= (1+1)c_1 + (3-1+1)c_0 = 2(4) + 3(1) = 11 \\ d_2 &= (1+2)c_2 + (3-2+1)\,c_1 = 3(1) + 2(4) = 11 \\ d_3 &= c_2 = 1 \end{aligned}$$

and hence $Q_4 = 1 + 11x + 11x^2 + x^3$.

Application 3. Another variation of this method is used to evaluate a power series whose coefficients are the Fibonacci numbers, F_n, defined by

$$\begin{aligned} F_0 &= F_1 = 1 \\ F_n &= F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2\,. \end{aligned}$$

The closed form of the series $f(x) = \sum_{n=0}^{\infty} F_n x^n$, when it converges, can be found by examining $f(x) + xf(x)$. One can show that

$$f(x) + xf(x) = \frac{f(x) - 1}{x}\,,$$

and hence

$$f(x) = \frac{1}{1 - x - x^2}\,.$$

Reference

Segal, A. C. (1983). Closed-form formulas for quasi-geometric series. *Two Year College Mathematics Journal*, 14(2), 118-122.

MT 3. Who came up with the term "Integral"?

Our reputation as a mathematics publisher
is based upon the unique talents of our authors . . .

Gerald L. Alexanderson • John D. Baley • Raymond A. Barnett • Edwin Beckenbach • Eric Crane Brody • Fred Buckley • Robert A. Carman • Thomas Cromer • Jacqueline M. Dewar • Irving Drooyan • Mark Eastman • Hugh M. Edgar • Wade Ellis, Jr. • Kevin Evans • Dale Ewen • George Feissner • Bernard Feldman • Norman Finizio • Mark Finkelstein • William Finzer • Katherine Franklin • Richard Fritz • Leon Gerber • Judith L. Gersting • Jimmie Gilbert • Linda Gilbert • Gary Gordon • Michael D. Grady • Stanley I. Grossman • Frank Gunnip • John Hardy • Linda Hawley • Steven Heath • Lee T. Hill • Abraham P. Hillman • Leonard I. Holder • J Martin Holstege • James F. Hurley • John Jobe • C. L. Johnston • Gail Jones • Lester T. Jones • Leonard M. Kennedy • Richard Kuechle • Gerasimos Ladas • Gloria Langer • Jeanne Lazaris • Walter Leighton • Ed Lodi • Vicky Lymbery • Charles P. McKeague • Elizabeth McMahon • Abshalom Mizrahi • John C. Molluzo • Jeff Morgan • C. Robert Nelson • Peter V. O'Neil • Thomas O'Neil • Sergei Ovchinnikov • Charles Paskewitz • Nancy Thies Pazner • Gilbert Perez • Gary Peterson • R. C. Pierce, Jr. • Douglas F. Riddle • Fred Safier • Hal M. Saunders • James T. Sedlock • F. Dennis Sentilles • Ken Seydel • James Spencer • Greg St. George • Mary Jo Steig • Raymond Southworth • Ron Staszkow • Michael Sullivan • W. Jene Tebeaux • Steve Tipps • Richard Tucker • Julian Weissglass • Dennis Weltman • David V. V. Wend • Alden Willis • Keith Wilson • Ellen T. Wood • William Wooton • Warren S. Wright • Dennis G. Zill

our commitment to quality throughout
the curriculum . . .

Covering the entire spectrum of undergraduate mathematics: Developmental • Precalculus • Applied Mathematics • Calculus • Topics in Advanced Mathematics

and our dedication to service!

Named "Publisher of the Year" by the National Association of College Stores

Wadsworth Publishing Company
10 Davis Drive • Belmont, CA 94002-3098 • 415/595-2350

W A D S W O R T H

FIGURE ON SUCCESS

Scott, Foresman Mathematics equals success in your classroom. It all adds up: respected authors, meticulous text development, innovative software, and conscientious service for you and your students.

Our successful core of remedial and precalculus texts is now augmented by a versatile list that continues to grow in size and scope. Each year, we offer more outstanding texts and supplements for your precalculus, calculus, and advanced courses.

This recent growth hasn't changed one very important thing—our fundamental commitment to quality. That's why we'll always welcome your inquiries, feedback, and publishing proposals. We figure our success depends on you.

SCOTT, FORESMAN AND COMPANY

SHORT COMMUNICATIONS

Matrices Which Are Equal To Their Own Adjoints

By

Dr. David Horowitz
Department of Mathematics
Golden West College
Huntington Beach, CA 92647

and

Mr. Nguyen Tuan Q.
13812 Magnolia Street
Garden Grove, CA 92644

In introductory linear algebra courses students are often assigned the exercises of constructing nontrivial $n \times n$ matrices which are nilpotent ($A^2 = O$) or idempotent ($A^2 = A$). However, one rarely tackles the problem of finding matrices which are equal to their own inverses ($A^2 = I$) or the related problem of finding matrices which are equal to their own adjoints ($\text{adj}(A) = A$). These latter matrices would have fascinating algebraic features which are immediate consequences of well-known classical adjoint properties (as shown in the chart).

CLASSICAL ADJOINT PROPERTIES for general matrices A & B	ALGEBRAIC PROPERTIES for A = adj(A) & B = adj(B)
RULE #1 $\text{adj}(A)\cdot\text{adj}(B) = \text{adj}(BA)$	$AB = \text{adj}(BA)$
RULE #2 $[\text{adj}(A)]^T = \text{adj}(A^T)$	$A^T = \text{adj}(A^T)$
RULE #3 $\lvert\text{adj}(A)\rvert = \lvert A\rvert^{n-1}$	$\lvert A\rvert = -1, 0,$ or 1 if $n > 2$
RULE #4 $A\cdot\text{adj}(A) = \text{adj}(A)\cdot A = \lvert A\rvert I$	$A^2 = \lvert A\rvert I$ (diagonal)

Thus, for the sake of algebraic curiosity and pedagogical completeness, it would be interesting to investigate matrices having the property

$$\text{adj}(A) = A.$$

When our linear algebra class first tackled the problem of constructing matrices which were equal to their own adjoints, one of the things we quickly discovered was that the only 2×2 examples are scalar multiples of the identity matrix:

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

For higher dimensions we had to restrict our investigations to non-singular matrices, because if A is singular then rank (adj(A)) < rank (A) (Eves, 1980, p. 155), and therefore $\text{adj}(A) \neq A$.

The equations for the 3×3 dimension were more cumbersome, but when we considered upper-triangular matrices they simplified considerably. The following matrix forms are equal to their own adjoints for all x and y:

$$\begin{pmatrix} 1 & x & y \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & x & xy-2 \\ 0 & 1 & y \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

In fact, it can be shown that these (along with the zero and identity matrices) are the *only* (real) 3×3 upper-triangular matrices which are equal to their own adjoints. (Their

transposes naturally provide lower-triangular examples.) Non-triangular examples include:

$$\begin{pmatrix} 0 & 0 & z \\ 0 & -1 & 0 \\ z & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & z \\ 0 & z & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & z & 0 \\ z & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $z = \pm 1$.

At this point the following characterization of all non-singular matrices which are equal to their own adjoints was discovered:

Theorem: If A is a non-zero matrix of order $n > 2$, then $A = \text{adj}(A)$ if and only if one of the following conditions is true:

Condition 1: $A^2 = I$ and $|A| = 1$.
Condition 2: $A^2 = -I$ and $|A| = -1$.

Condition 2 can only occur if n is an even integer greater than 2.

Proof: To establish the necessity of Conditions 1 or 2, assume that $A = \text{adj}(A)$. Since A is invertible, RULE #3 implies

$$|A| = \begin{cases} \pm 1 & \text{if } n \text{ is even} \\ +1 & \text{if } n \text{ is odd.} \end{cases}$$

Moreover by RULE #4, we have

$$A^2 = \begin{cases} \pm I & \text{if } n \text{ is even} \\ I & \text{if } n \text{ is odd .} \end{cases}$$

To prove the sufficiency of Condition 1, observe that

$$A = A^{-1} = |A|A^{-1} = \text{adj}(A).$$

Similarly, to prove the sufficiency of Condition 2, observe that

$$A = -(-A) = -(A^{-1}) = |A|A^{-1} = \text{adj}(A). \quad \blacksquare$$

Using this characterization it is possible to exhibit higher-dimensional examples. If n is an odd integer ≥ 3 then

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & \cdots & x_n \\ 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

is equal to its own adjoint for all $x_2, x_3, \ldots, x_n$.

If n is an even integer ≥ 4 then

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & \cdots & x_n \\ 0 & -1 & -2x_3/x_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

is equal to its own adjoint for all $x_2, x_3, \ldots x_n (x_2 \neq 0)$. The only differences between these two forms lie in the third column.

Reference

Eves, H. (1980) *Elementary matrix algebra*. New York: Dover Publications, Inc.

How Many Final Games To Two Consecutive Championships?

by Joseph F. Stokes
Department of Mathematics
Western Kentucky University
Bowling Green, Kentucky 42101

It is not unusual for teams in the championship game of a basketball tournament to be given near equal chances of winning. Let us go one step further and assume that each of the two teams which has not been eliminated prior to the championship game always has an equal chance of winning the tournament. What is the expected number of tournaments in which a given team must be in the final game to win two consecutive championships?

Under the stated conditions, it is easy to find the expected number of tournaments in which a given team must be in the final game to win its first championship. In this case, it is known that all trips have resulted in losses until the first win. If we use s and f to denote success and failure, respectively, then we can write the various possible outcomes as s, fs, ffs, fffs, etc.

Let n be the number of trips and p and qbe the probability of a success and failure, respectively. Therefore, $E(n) = \sum_{n=1}^{\infty} nq^{n-1}p = p + 2qp + 3q^2p + 4q^3p + 5q^4p + \cdots$.

This may be rewritten as:

$E(n) = (p) + (qp + qp) + (q^2p + q^2p + q^2p) + (q^3p + q^3p + q^3p + q^3p) + \cdots$,

or

$E(n) = (p + qp + q^2p + q^3p + q^4p + \cdots)$ (Sum of first terms within each set of parenthesis.)
$+ (qp + q^2p + q^3p + q^4p + \cdots)$
$+ (q^2p + q^3p + q^4p + \cdots)$
$+ (q^3p + q^4p + q^5p + \cdots)$
$+ (q^4p + q^5p + q^6p + \cdots)$
etc.

or

$E(n) = \frac{p}{1-q} + \frac{pq}{1-q} + \frac{pq^2}{1-q} + \cdots$ because each of the above series is geometric.

Since the last given expression for E(n) is also geometric, its sum is given by

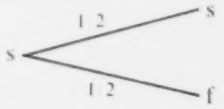
$$\frac{1}{1-q} \cdot \frac{p}{1-q} = \frac{p}{(1-q)^2}.$$

Thus, when $p = q = 1/2$,

$$E(n) = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2.$$

Examples of possibilities for the first two consecutive wins are ss, sfsffss, sfffsfsffss. This differs from the problem of the first win because no pattern whatsoever can be established. Hence, the aforementioned method is of no value to us.

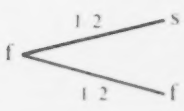
Fortunately, we can apply the following recursion approach. Let a be the expected number of championship games in a sequence beginning with s and ending with the first ss. Let b be the expected number of championship games in a sequence beginning with f and ending with the first ss. All possible beginnings are represented by the followng two tree diagrams:



This tree diagram begins with s. Therefore, by definition, the expected number of games in a sequence ending with the first ss is a. The probability is 1/2 for each of the two branches to occur. If the upper branch were to occur, then the sequence is ss which represents only two final games. If the lower branch were to occur, there are b games in the sequence beginning with the f. One game has been played prior to the f; hence, the number is b + 1. In equation form, this becomes

$$a = \tfrac{1}{2}(2) + \tfrac{1}{2}(b + 2).$$

Likewise, the following equation is the result of the tree diagram beginning as indicated.



$$b = \tfrac{1}{2}(a + 1) + \tfrac{1}{2}(b + 1)$$

Simplifying these two equations, we obtain:

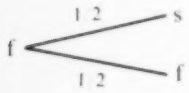
$$2a = 3 + b \text{ and } b = 2 + a.$$

Hence, $a = 5$, $b = 7$. Therefore, the expected number of games in a sequence ending with the first ss is given by:

$$E(n) = \tfrac{1}{2}a + \tfrac{1}{2}b = \tfrac{1}{2}(5) + \tfrac{1}{2}(7) = 6$$

When shown the method, the first reaction of a colleague of mine was that it could not be done so simply. He simulated 20,000 such sequences and found a mean of 5.99955.

It might be of interest to apply the recursion method to the aforementioned problem of the first win. To do so, let a be the expected number of games in a sequence beginning with a win. Likewise, let b be the expected number of games in a sequence beginning with a failure and ending with the first win.



$$a = 1$$

$$b = \tfrac{1}{2}(a + 1) + \tfrac{1}{2}(b + 1)$$

Therefore, $2b = a + 1 + b + 1$, or $b = a + 2$. Since $a = 1$, we have $b = 3$. Therefore,

$$E(n) = \tfrac{1}{2}(1) + \tfrac{1}{2}(3) = \tfrac{1}{2}(4) = 2.$$

Observe that this is much simpler than the traditional method presented earlier in this paper.

References

Johnsonbaugh, R. (1984). *Discrete mathematics*. New York: Macmillan.

Mansfield, E. (1986). *Basic statistics with applications*. New York: Norton.

Mendenhall, W. (1983). *Introduction to probability and statistics* (6th ed.). Boston: PWS Publishers.

Mood, A. & Graybill, F. (1963). *Introduction to the theory of statistics*. New York: McGraw-Hill.

CALL FOR ARTICLES

We are presently on the threshold of a new decade. As a way of welcoming the 1990s, the staff of *The AMATYC Review* is planning to publish a special issue that will focus on emerging innovative programs in two-year college mathematics education.

With this purpose in mind, you are invited to submit articles that forecast and, perhaps, will guide mathematics education in the 1990s. Articles should be wide ranging in focus and may be more inclusive than those that usually appear in this journal. In addition, authors should provide insight into the forces that will affect the changes in our present programs.

The deadline for the submission of manuscripts is February 1, 1989. The publication date for the special issue is October 1, 1989. For further details contact the editor.

BEST COPY AVAILABLE

The Ohio State University Calculator and Computer Precalculus Project: The Mathematics of Tomorrow Today!

by Franklin Demana and Bert K. Waits
The Ohio State University
Department of Mathematics
231 W. 18th Ave.
Columbus, Ohio 43210

Franklin Demana received his Ph.D. in Mathematics from Michigan State University in 1966. He is currently Associate Professor of Mathematics at The Ohio State University where he has been on the faculty since 1966. He is the Director of Freshman Mathematics at Ohio State. Dr. Demana is the author of many articles in professional journals and the coauthor of two mathematics textbooks. His current areas of interest are in mathematics curriculum development at the middle school through calculus level utilizing calculators and computers. Bert K. Waits received his Ph.D. in Mathematics Education from The Ohio State University in 1969. He is currently Associate Professor of Mathematics at The Ohio State University where he has been on the faculty since 1965. He is the originator and Director of The Ohio Early College Mathematics Placement Testing Program operated for the Ohio Board of Regents. Dr. Waits is the author of many articles in professional journals and is the coauthor of four mathematics textbooks. His current areas of interest are in actuarial mathematics and mathematics curriculum development at the precalculus and calculus level utilizing calculators and computers.

Fundamental concepts of function, graphing, solving equations, and solving inequalities are crucial for subsequent work in advanced mathematics and related applied areas. For example, The College Board (1985, p. 29) states that

> The concept of function is central to mathematics, and students entering college not only need to understand what functions are in general but also to be familiar with examples . . . They should be able to use computers and other tools to represent functions graphically.

There is strong evidence from the Second International Mathematics Study (SIMS) that suggests traditional methods and textbooks used in many of today's United States high school precalculus classrooms do not adequately prepare students for collegiate work. SIMS pretest and posttest average scores for United States precalculus twelfth graders on standard items concerned with graphing and analytic geometry were 34% and 43%, respectively. A national report on the SIMS study (McKnight et al, 1987, p. vii) states that

> The achievement of the U.S. precalculus students (the majority of twelfth grade college-preparatory mathematics students) was substantially below the international average. In some cases the U.S. ranked

with the lower one-fourth of all countries in the Study, and was the lowest of the advanced industrialized countries.

Data about students at The Ohio State University, a typical open admissions public university, show that less than one in six high school students entering the university with *four or more years of college preparatory mathematics* are ready to begin their study of collegiate mathematics with calculus (Demana and Waits, 1988). Demana and Waits (in press, a) give data strongly suggesting that the portion of college freshmen prepared to begin their mathematical studies with calculus must be increased if the number of students majoring in engineering and science is to be increased.

This paper describes a nontraditional approach to teaching precalculus mathematics that makes use of calculators and computers to accurately graph functions and relations and to solve a wide variety of "difficult" but "interesting and realistic" problems. Microcomputers and graphing calculators (really pocket computers) are powerful problem solving devices that permit students to make and test generalizations by looking at a large number of examples in a short period of time. Technology can dramatically change the way we teach mathematics and the way students learn mathematics. The standard traditional approach uses arithmetic and algebraic information to produce graphs of functions and relations. Now, accurate computer graphs can be used to determine the behavior of functions and relations and to develop geometric intuition important in the study of calculus and advanced mathematics.

The new approach is currently being class tested in The Ohio State University Calculator and Computer Precalculus (C^2PC) Project[1]. This project is a cooperative precalculus curriculum development effort of three Public School Districts in Ohio (Columbus, Upper Arlington and Southwestern), the Mathematics Department and the Educational Theory and Practice Department at The Ohio State University. The major goals of the C^2PC Project are:

1. To create instructional materials that make effective use of computer based graphing and strengthen student problem solving skills.
2. To improve student understanding of functions and related graphical concepts, domains of mathematical deficiency in the current college preparatory curriculum.
3. To significantly increase the number of students adequately prepared to pursue higher education in mathematics, science, and technical fields.

The new precalculus course is intended for both high school and college students who have successfully completed Algebra I, Geometry, Algebra II and includes most of the traditional topics covered in a post Algebra II functions, trigonometry and analytic geometry course. Problem solving is emphasized and real world applications are integrated throughout the materials. Important algebraic skills and techniques are included. Beside providing a new approach, computer graphing also provides a geometric interpretation and a check for solutions found algebraically.

The project staff have developed easy to use, interactive computer graphing software (Demana and Waits, in press, c) for the Apple II (+ / e / c /GS), IBM, and Macintosh computers that allows the student to obtain, in seconds, the graph of any function, conic equation, parametric equation or polar equation. Students view the portion of the graph

[1] The C^2PC Project is partially funded by the Ohio Board of Regents, the Standard Oil Company, and the National Science Foundation. The views expressed are those of the authors and do not necessarily reflect the views of the funding agencies.

that appears in the rectangular region $L \leq x \leq R$ and $B \leq y \leq T$ by inputting the values of L, R, B, and T. This rectangular region is called the *viewing rectangle* $[L, R]$ by $[B, T]$. The graphing programs used with the project textual materials have two important features. The first, called *zoom out*, is very much like using the computer to place the user at large distances from a graph so that more of it can be observed at one time. Zoom out is a process that allows the user to start with a graph in a given viewing rectangle and experiment to determine the graph outside of the viewing rectangle. Once the graph inside and outside of a viewing rectangle is known, we say we have determined the "complete graph." The second feature, called *zoom in*, uses the computer to put a portion of a graph under a very powerful microscope. Zoom in allows the user to solve traditionally inaccessible equations and inequalities with a high degree of accuracy. Students use these interactive features to make graphing a very effective problem solving tool.

Relatively inexpensive graphing calculators with features such as zoom in and zoom out (e.g., the Casio *fx-7000G*) are used in place of, or in tandem with, microcomputers to obtain accurate graphs. Such calculators provide graphing devices needed for homework and ease the logistics of computer lab scheduling. Most of the project students are using the Casio *fz-7000G* or *fx-8000G* graphing calculators as well as computers. We have found that most students are remarkably proficient at using these devices for graphing functions after three to four days of instruction. The instruction can be handled in context, so textual material is also being covered at the same time.

The project textual materials (Demana and Waits, in press, b) do not require the use of a particular computer or graphing calculator. Students only need frequent access to a device that has an effective graphing utility. The three examples that follow illustrate how computer graphing is used in the actual project textual material with precalculus students.

Example 1. Pam and Jerry's budget will allow monthly mortgage payments of $700. Assume they can obtain a 25 year mortgage at some fixed rate of annual interest, called the APR rate. What APR rate would Pam and Jerry need if they wanted to purchase a home requiring a $65,000 mortgage?

Solution. This is an example of an annuity where the present value (amount borrowed), term (300 months), and monthly annuity payment ($700) are known. Annuities provide an excellent application of finite geometric series. The present value, as a function of the *monthly* interest rate x, is given by $P(x) = \frac{700(1 - (1 + x)^{-300})}{x}$ (Waits, 1979). Any point (x_0, y_0) common to the graphs of the function $y = P(x)$ and the line $y = 65{,}000$ determines a solution to the problem with APR interest rate equal to $12x_0$. We restrict the values of x to $0 \leq x \leq 1$ because the monthly interest rate is positive and very likely less than 100% per month. Because our graph must include the point of intersection of the line $y = 65{,}000$ with $y = P(x)$, the values of y are restricted to $0 \leq y \leq 80{,}000$. Figure 1 gives the graphs of $y = P(x)$ and $y = 65{,}000$ in the viewing rectangle [0, 1] by [0, 800, 000] determined by $0 \leq x \leq 1$ and $0 \leq y \leq 80{,}000$.

It is impossible to read the coordinates of the point of intersection with much accuracy from the graph in Figure 1. However, we can see that x is less than 0.1, so the next viewing rectangle is chosen to be [0, 0.1] by [0, 80,000] (Figure 2).

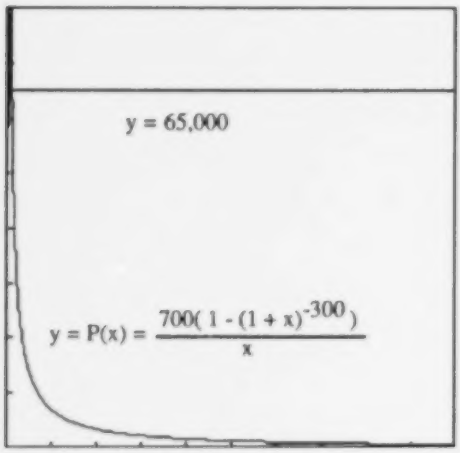


Figure 1

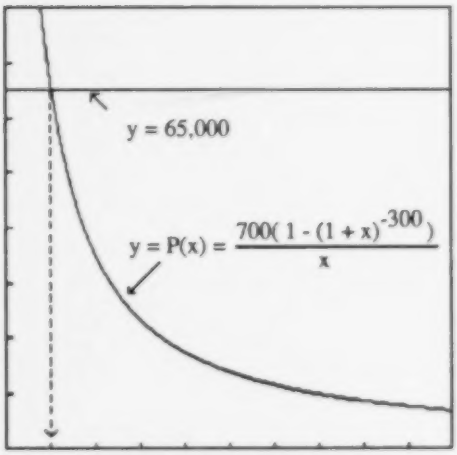


Figure 2

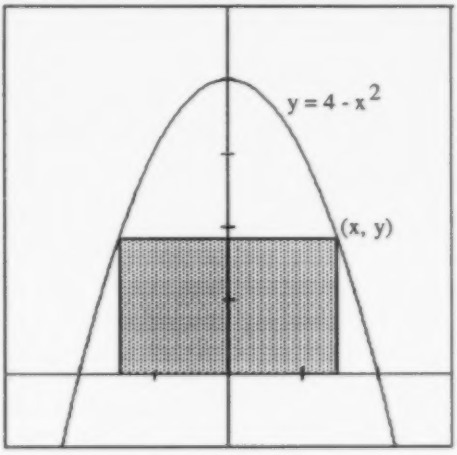
Figure 2 shows that the desired monthly interest rate is about 0.01. Thus, the APR is about 0.12 or 12%. The actual answer accurate to 8 decimal places is 0.12319696. This answer can be found graphically using a zoom in method which is illustrated in the next example.

The graph in Figure 2 indicates that the function P is decreasing for $x > 0$. This is reasonable because, with a fixed mortgage payment, the possible loan amount (mortgage) decreases as the interest rate increases.

Therefore, there is one and only one simultaneous solution to the pair of equations $y = P(x)$ and $y = 65{,}000$ for $x > 0$.

A traditional calculus exercise that is accessible to precalculus students with the aid of technology is illustrated in Example 2.

Example 2. Find the maximum possible area of a rectangle that has its base on the x-axis and its upper two vertices on the graph of the equation $y = 4 - x^2$ (Figure 3).

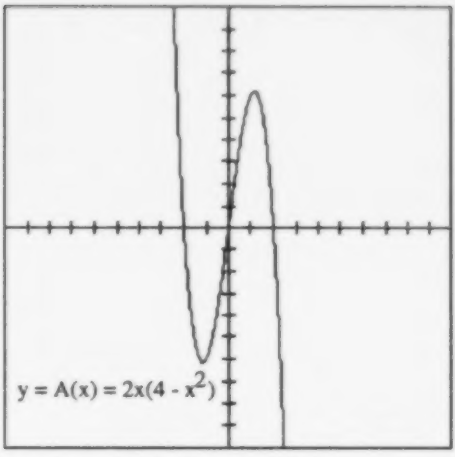


[-3, 3] by [-1, 5]

Figure 3

Solution. Let the coordinates of the upper right-hand corner of the rectangle be (x, y). The width of the rectangle is $2x$ and the height is y. Because the point (x, y) lies on the graph of $y = 4x^2$ the height y of the rectangle is $4 - x^2$ and the area of the rectangle is $A(x) = 2xy = 2x(4-x^2)$. The only values of x that give possible rectangles are $0 < x < 2$. (Why?) The graph of $y = A(x) = 2x(4 - x^2)$ given in Figure 4 shows that A has a local maximum value between $x = 0$ and $x = 2$. Because A represents the problem situation only for values of x between 0 and 2, this local maximum value of A must be the solution to the problem. Figure 4 shows that the maximum function value between $x = 0$ and $x = 2$ is about 6 and occurs for x about 1.2.

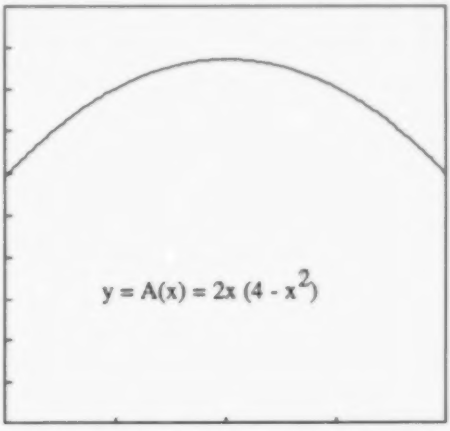
With a given graph on the screen, students can use a zoom in process to quickly redraw the graph in a new, smaller "zoom in" rectangle containing a given point or region of interest. They literally *draw* the new rectangle on the screen. This process allows students to create a nested sequence of rectangles that decrease in size and converge to a desired point in the original viewing rectangle. Applying the zoom in process



[-10, 10] by [-10, 10]

Figure 4

several times around the local maximum between 0 and 2 in Figure 4 results in the graph shown in Figure 5. This figure reveals that the value of x that produces the maximum area of the rectangle is approximately 1.1547, and the corresponding maximum area A is about 6.15840286. The error in the observed maximum area is less than 0.0000001.



[1.1545, 1.1549] by [6.158402, 6.158403]

Figure 5

Zoom in is a very powerful problem solving strategy and leads in a very natural way to a discussion of error and accuracy. Students can use zoom in to solve equations, inequalities and problems with accuracy up to the limits of machine precision that rivals

sophisticated numerical techniques. And these equations, inequalities, and problem situations need not be contrived, and can be very complicated.

Example 3. Describe the behavior of $f(x) = \dfrac{2x^4 + 3x^3 + x^2 + 2}{x^3 - x^2 + 20}$.

Solution. Project students start out with the graph of f in the standard viewing rectangle $[-10, 10]$ by $[-10, 10]$ (Figure 6). This graph indicates that f probably has one vertical asymptote between $x = -3$ and $x = -2$. End behavior, the behavior of f for $|x|$ large, is determined by viewing the graph in several large viewing rectangles. This is accomplished quickly using a process called zoom out. If the graph of a function in the viewing rectangle $[L, R]$ by $[B, T]$ is on the screen, then for any positive number k, we have

(a) Zoom out *horizontally* redraws the graph in the viewing rectangle $[kL, kR]$ by $[B, T]$;
(b) Zoom out *vertically* redraws the graph in the viewing rectangle $[L, R]$ by $[kB, kT]$; and
(c) Zoom out *both* redraws the graph in the viewing rectangle $[kL, kR]$ by $[kB, kT]$.

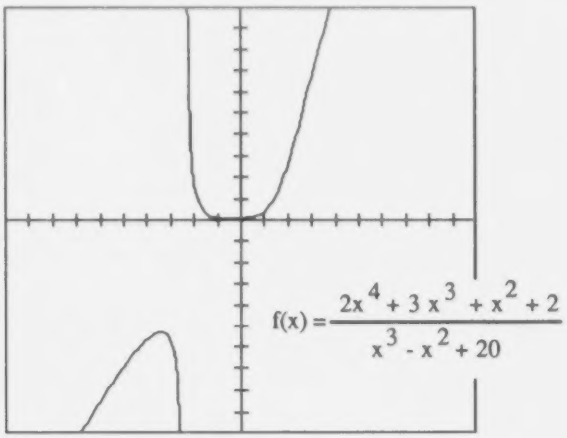
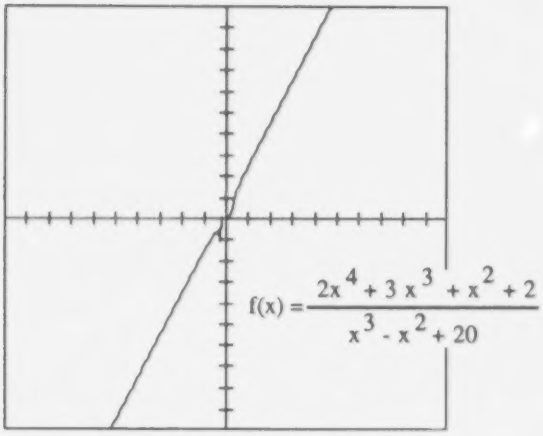


Figure 6

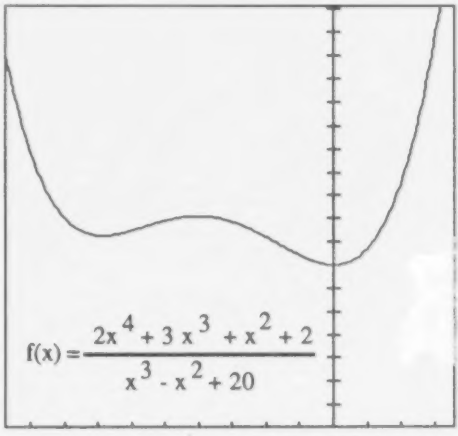
Zoom out both in Figure 6 with $k = 10$ produces the graph in the $[-100, 100]$ by $[-100, 100]$ viewing rectangle shown in Figure 7. The behavior of f near $x = 0$ is lost, but the end (limiting) behavior of f appears to be a line with slope 2. This can be confirmed algebraically by using long division to write $f(x) = 2x + 5 + \dfrac{6x^2 - 40x - 98}{x^3 - x^2 + 20}$. As $|x|$ approaches ∞, $f(x)$ approaches $2x + 5$. The line $y = 2x + 5$ is called a *slant asymptote*. If the graph of $y = 2x + 5$ is overlaid in Figure 7, it would virtually be impossible to distinguish between the two graphs.



[-100, 100] by [-100, 100]

Figure 7

It is interesting to note that this rational function has some interesting behavior near the origin that is "hidden" from view in Figure 6. Two applications of zoom in around the lower portion of the right-hand branch of f in Figure 6 reveal the three local extrema shown in Figure 8. (Are there other hidden extrema?)



[-0.9742, 0.3618] by [0.0930056, 0.111153]

Figure 8

Computer graphing is a powerful tool for mathematical exploration and is an important concept and intuition builder. However, Example 3 shows that computer graphing will not replace the need for the formal study of calculus because *predicting* the "hidden" behavior of f exhibited in Figure 8 is impossible without calculus.

The C^2PC computer based approach to precalculus mathematics instruction has several important benefits.

(1) It takes advantage of what the computer does best: namely, fast computation with outstanding graphics.

(2) It is interactive and expects students to be active partners in the learning process by using the computer as an investigative "tool" and powerful general problem solving device.

(3) It provides a focus on functions and relations and their graphs, important fundamental concepts.

(4) It makes generalization possible for students by taking advantage of the speed and power of computers to investigate and study many more examples and problems than possible using limited traditional methods.

(5) It provides students with a thorough understanding of the behavior of a significantly expanded class of functions and relations through a graphical approach.

(6) It gives students the opportunity to establish the connections between problems, equations describing the problems, and graphs of the equations describing the problem.

The 1987-88 C^2PC project team consist of mathematics educators from both college and high school: Pam Dase and Patricia Borys from Centennial High School, Bruce Blackston and Dan Rohrs from Upper Arlington High School, Ron Meyer from Franklin Heights High School, Fred Koenig from Walnut Ridge High School, Frank Demana and Bert Waits from the Department of Mathematics at The Ohio State University, Alan Osborne and Greg Foley from the College of Education at The Ohio State University, and graduate students Chris Browning, Amy Edwards, Bishnu Naraine, Sandy Davey, Linda Taylor, Ann Farrell, and Laurie Wern.

The C^2PC precalculus materials will be field tested in colleges (including The Ohio State University) as well as in high schools during the 1988-89 academic year. The materials provide for a year long high school precalculus course or a two quarter college algebra and trigonometry sequence. Some of the C^2PC ideas and approaches have been piloted in college algebra classes at The Ohio State University during the past two years.

There are several crucial educational issues under investigation in the C^2PC project. Which traditional topics should receive reduced attention to provide time for a computer graphing approach? What is the role of paper and pencil techniques for graphing and problem solving in light of technology? How will traditional algebraic skill be effected? What algebraic skills are necessary for the "Calculus of the twenty-first century"?

We expect the increased attention to graphing in the college algebra portion of the course to reduce the time needed with transcendental functions. Deeper understanding about underlying algebraic ideas is expected because of the natural interplay between algebraic and geometric representation of situations given by the new approach. Less time spent on algebraic techniques is not expected to cause loss in algebraic skill because of the integrated geometric approach.

We invite colleagues at other colleges to use the C^2PC materials in their Algebra and Trigonometry classes. Interested readers can obtain more details about the C^2PC project, computer software, textual materials, and pilot and field test evaluation data, by writing the authors (Mathematics Department, 231 W. 18th Ave., Columbus, Ohio 43210).

References

College Entrance Examination Board, (1985). *Academic preparation in mathematics: Teaching for transition from high school to college.* New York, NY: CEEB.

Demana, F., & Waits, B.K. (1988). Is three years enough? *The Mathematics Teacher, 81,* 11-14.

Demana, F., & Waits, B.K. (in press, a). Entering mathematics skills and success in college. *The School Counselor.*

Demana, F., & Waits, B.K. (in press, b). *Precalculus mathematics - A computer graphing approach.* Addison-Wesley.

Demana, F., & Waits, B.K. (in press, c). *Master Grapher,* [Computer software]. Addison-Wesley.

McKnight, C.C., Crosswhite, F.J., Dossey, J.A., Kifer, E., Swafford, J.O., Travers, K.J., & Cooney, T.J. (1987). *The underachieving curriculum: Addressing U.S. school mathematics from an international perspective.* Champaign, Illinois: Stipes Publishing.

Waits, B.K. (1979). *"The mathematics of finance revisited through the hand calculator." In Applications of mathematics,* 1979 Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council.

CALL FOR REFEREES

Readers are invited to help the editor to evaluate manuscripts by becoming members of *The AMATYC Review* reviewer panel. Write to Don Cohen, SUNY Ag-Tech College, Cobleskill, NY 12043 for more information.

MT 4. Who came up with the integral symbol back in the mid-seventeenth century?

Taste Success

The New HBJ College Outlines Offer a Fresh Approach to Studying Mathematics

Available in these Mathematics titles:

Calculus
College Algebra
Analytic Geometry
Trigonometry
Introductory Algebra
Intermediate Algebra
Statistics I
Statistics II
Arithmetic
Applied Fourier Analysis
Applied Vector Analysis

Toll Free Ordering
1-800-543-1918
In MO and AK, call collect: 314-528-8110

HBJ Harcourt Brace Jovanovich, Inc., 1250 Sixth Ave., San Diego, CA 92101-9803

Using Spreadsheets to Offer a More Realistic Business Mathematics Course

by Edward D. Laughbaum
Columbus State Community College
Mathematics Department
Columbus, OH 43215

Ed has taught mathematics at Columbus State Community College for the past 17 years. He received his BS in 1967 from Otterbein College and his MS in 1971 from The Ohio State University.

Business mathematics in the classroom should be very much like actual mathematics in business. Typically, business mathematics is done with a computer and software. Both of these tools can easily be used in the business math classroom or a computer lab. The software is available and is within the budget limitations of many schools and most schools have a computer lab. Thus we need to move in the direction of integrating the two worlds.

The most popular business software is Lotus 1-2-3. Because of this popularity, there is a good likelihood that your business math students will use this spreadsheet software in business. Many colleges are using Lotus, not just in the business departments, but in mathematics, physics, and other technical areas. In a business math class you may not want to take the time to teach Lotus 1-2-3; however, if you can find the time to teach only two to three hours of 1-2-3, your students will be able to use 1-2-3 and understand its potential in the business world.

The student needs to learn the following tasks:

1. How to turn the equipment on and off.
2. How to enter and exit 1-2-3.
3. How to move the pointer around the screen.
4. How to enter data in cells.
5. How to load a file.
6. How to execute a macro.
7. How to handle diskettes properly.

The instructor who teaches the class need only know a little more about 1-2-3. For example, when a student strikes the wrong key at the wrong time, the instructor will need to be able to get 1-2-3 back to ready mode.

Many of the business math topics such as payroll, taxes, trade and cash discounts, markup, markdown, inventory, simple and compound interest, annuities, depreciation, and descriptive statistics can be put on a spreadsheet using only one screen by scaling down their sizes. By putting the entire visible worksheet on one screen, it is less confusing to the beginning student yet the basic concepts of the topic have been preserved. The formatting of the cells, the sample data, and the proper formulas for each topic can be put on each worksheet so that the student need only change selected data in order to answer questions of the "what if" nature. So that the student can do more difficult tasks without learning more about Lotus, complicated tasks can be put in macros. For example, to get a hard copy of a worksheet would only require the use of two keys (like Alt-p). To exit 1-2-3, the student may need only to type Alt-q.

By having the worksheet prepared in advance, as well as having taught the mathematics before the worksheet is assigned, the student can concentrate on the use of the spreadsheet program as a method of solving business problems. Once the worksheets have been finished and saved on a disk, the next step is to ask questions of the student that will demonstrate the value of a spreadsheet program. The instructor can also prepare questions that can demonstrate the mathematics taught. The student can answer the questions during class or as an outside assignment. Eight to ten questions may take from 15 minutes to 60 minutes depending on the student. The intent of such a venture is not to make spreadsheet software part of the business math curriculum, but to use a proven business math tool to help teach business math. This is not to suggest that it could not be part of the curriculum. It may be decided department-by-department. Because the business math student is usually a beginner in using software, there is no great rush to have the student learn Lotus 1-2-3. The goal is to *use* Lotus 1-2-3. The learning of Lotus 1-2-3 perhaps should be left for other courses.

In the Mathematics Department at Columbus State Community College, we have been using spreadsheets in our business math classes for nearly three years. During the seventh week of Winter Quarter 1987, 172 students in business math were surveyed on the use of the computer and the computer materials. Seventy five (75) percent had never used a spreadsheet before. The survey showed that 83% of the students felt that business math was more interesting because of the computer usage. Eighty (80) percent indicated that the computer materials helped them understand how computers are used in business. Fifty one (51) percent thought that the computer usage helped them learn business math. Seventy nine (79) percent thought that the computer materials would help them in other courses and 69% thought it would help when they got a job. Finally, 61% said they would like to know more about spreadsheets.

In summary the basic idea behind the using of Lotus 1-2-3 is that students should be introduced to sophisticated professional software in college so that business mathematics is much like the mathematics of business. The following is a typical spreadsheet exercise.

LAB 4: MARKUP SPREADSHEET

ITEM	COST	RATE c	RATE s	MARKUP	SELLING
MONITOR	$76.00	200	66.7	$152.00	$228.00
CPU	$600.00	50	33.3	$300.00	$900.00
MODEM	$175.00	400	80.0	$700.00	$875.00
PRINTER	$156.00	60	37.5	$93.60	$249.60
KEYBOARD	$100.00	150	60.0	$150.00	$250.00
	$1,107.00			$1,395.60	$2,502.60
AVERAGE RATE based on COST =			126.1%		

ITEM	COST	RATE s	RATE c	MARKUP	SELLING
MONITOR	$76.00	20	25.0	$19.00	$95.00
CPU	$600.00	35	53.8	$323.08	$923.08
MODEM	$175.00	30	42.9	$75.00	$250.00
PRINTER	$156.00	40	66.7	$104.00	$260.00
KEYBOARD	$100.00	60	150.0	$150.00	$250.00
	$1,107.00			$671.08	$1,778.08
AVERAGE RATE based on SALES =			37.7%		

DIRECTIONS:

1. The top half of the spreadsheet shows markup based on COST. The only values that can be changes are the RATE c and cost.
2. The bottom half shows markup based on SALES. The only values that can be changed are the RATE s and cost.
3. To load this lab type /fr — then highlight lab 4.
4. If you do not successfully enter your name when you load the file, you must enter your Name on this worksheet by holding down the Alt key and typing the N key. Strike the return (enter) key after your name and again after your SS#.
5. To generate a hard copy of this spreadsheet, hold down the Alt key and type the P key.
6. To quit the session hold down the Alt key and type the Q key.

THE MATHEMATICS OF LAB 4

The selling price (S) of an item with a cost of (C) and a markup rate based on cost of (R), would be found by adding the markup (M) to the cost. That is:
$S = C + M$ where $M = CR$ or $S = C + M$ or $S = 1C + CR$ and finally $S = C(1 + R)$.
Therefore, to find the selling price when the cost and the markup rate are known, multiply the cost times 1 plus the rate of markup.

To find the selling price when the markup rate is based on the selling price is not quite so obvious. When rate is based on the selling price, we mean that $M = RS$; that is, markup equals rate times the selling price. To get the selling price when the cost and markup rate based on sales are known, we reason as follows:
$S = C + M$ where $M = RS$ or $S = C + M$ or $S = C + RS$ so $S - RS = C$ or $1S - RS = C$ and $S(1 - R) = C$ and finally $S = C / (1 - R)$.
Therefore, to find the selling price when the cost and the rate of markup based on sales are known, divide cost by 1 minus the rate based on sales.

LAB 4 QUESTIONS

1. Using only the top half of the worksheet, adjust the RATE c so that the total markup is as near to $4000 as you can make it. Make a hard copy of this spreadsheet.
2. Using only the bottom half, adjust the RATE s so that the total markup is as near to $4000 as you can make it. Make a hard copy on the same page as problem 1.
3. Adjust all RATE c in the top half so that the average RATE is approximately 200% (do not make all rates 200%). Adjust all RATE s in the bottom half so that the average is approximately 80% (do not make all rates 80%). Make a hard copy of this spreadsheet.
4. Using the top only, what would be the total markup if all items had a RATE c of 125% (give all 5 items a 125% markup). Answer the same question for 100%, 150%, 300%.
5. Change all of the RATE c to 400% in the top and change all of the RATE s in the bottom to 80%. Is the total selling price on top equal to the total selling price on the bottom? Why?
6. What happens if the RATE s is 100%? Why?
7. Re-load the original LAB 4 spreadsheet (Note the TOTAL price, TOTAL cost, and TOTAL markup.) Then increase the cost of each item by 10% on the top half of the spreadsheet. You will need a calculator for this. What is the *increase* in total selling price, total cost, and total markup?
8. Re-load the original LAB 4 spreadsheet. On the bottom half add $40 to the cost of each item. How much *increase* is there in the total selling price, total cost, and total markup?

BEST COPY AVAILABLE

LAB 4 ANSWERS

Name _____ Class _____ Hour _____

QUESTIONS:

1. On the hard copy, identify the answer by circling the answer and mark it as question 1. Attach it to this sheet.
2. Same as above.
3. Same as above.
4. 125% _____, 100% _____, 150% _____, 300% _____.
5. _____

   _____
6. _____
7. Increase in selling price _____ total cost _____

   total markup _____ .
8. Increase in selling price _____ total cost _____

   total markup _____ .

Fractions

I particularly like the following quote because I know without a doubt that fractions are the most difficult concept that I ever attempt to teach!

Jean Burr Smith
Middlesex Community College
Middletown, CT 06457

"A fraction is nothing more nor less than an unexecuted division: i.e. division indicated but not performed, the numerator being the dividend and the denominator the divisor.

This is the key to a knowledge of fractions, and this knowledge of fractions is in turn the key to higher arithmetic and algebra.

He who has the key intelligently in his possession will advance rapidly and pleasantly, while he who neglects the key will see no beauty in mathematics."

Treatise on Arithmetic: Combining Analysis and Synthesis
James Eaton, Philips Academy, 1857

MT 5. Which famous mathematician came up with the term "Mantissa" back in 1624?

New 1989 Titles from
HARCOURT BRACE JOVANOVICH, Inc.

BASIC MATHEMATICS AND PROBLEM SOLVING
Second Edition

BEGINNING ALGEBRA AND PROBLEM SOLVING
Second Edition

Both by Alan Wise

INTERMEDIATE ALGEBRA AND PROBLEM SOLVING
Second Edition

Alan Wise, Richard Nation, and Peter Crampton

COLLEGE ALGEBRA AND TRIGONOMETRY
Third Edition

Robert Ellis and Denny Gulick

COLLEGE ALGEBRA

Available Now!

TRIGONOMETRY

Both by Adelbert Hackert and Gene R. Sellers

APPLIED TRIGONOMETRY

Michael N. Payne and Wayne R. Anderson

HBJ

HARCOURT BRACE JOVANOVICH, Inc.
College Sales Office
7555 Caldwell Avenue, Chicago, IL 60648
(312) 647-8822

WEST'S MATH LINE

THE TOOLS FOR A SOLID UNDERSTANDING OF MATH.

At West, we want your students to not only learn, but to truly understand concepts of mathematics. That's why our authors take great pains to write texts that are thorough, comprehensive, and accurate. A complete supplement package accompanies each text to give students additional study tools, and to assist you in teaching your course.

So give your students the tools for a solid understanding of mathematics. Give them a text from West.

Elementary Algebra

Goodman and Hirsch
Understanding Elementary Algebra, 2e
NEW for 1989!

Intermediate Algebra

Hirsch and Goodman
Understanding Intermediate Algebra, 2e
NEW for 1989!

College Algebra

Barros-Neto
College Algebra with Applications, 2e
1988

Cohen
College Algebra, 2e
NEW for 1989!

Steinlage
College Algebra, 2e
1988

College Algebra and Trigonometry

Barros-Neto
College Algebra and Trigonometry with Applications, 2e
1988

Cohen
College Algebra and Trigonometry, 2e
NEW for 1989!

Steinlage
College Algebra and Trigonometry, 2e
1988

Trigonometry

Vancil
Trigonometry
1988

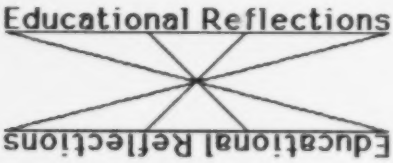
Discrete Math

Barnier and Chan
Discrete Mathematics with Applications
NEW for 1989!

Statistics

Kvanli
Statistics: A Computer Integrated Approach
1988

For more information on these and our complete line of mathematics texts, please write to:
West Publishing Company
College Division, 6F, 50 W. Kellogg Blvd., P.O. Box 64526, St. Paul, MN 55164



Let Us Teach Exploration

by Gregory D. Foley
The Ohio State University
Columbus, OH 43210-1172

Preparation of this article was supported in part by grants from British Petroleum (America), the National Science Foundation, the Ohio Board of Regents, and The Ohio State University. The opinions and conclusions herein expressed are those of the author and do not necessarily reflect the views of these funding agencies.

I thank the colleagues and associates who offered suggestions during the preparation of this article.

Polya said, "Let us teach guessing" (1950). This article attempts to expand on this worthwhile theme. It advocates our helping students view mathematics as a lively, experimental science in which they learn through the exploration of ideas and interrelationships. It advocates moving from a computational focus to a conceptual focus by exploiting modern technology to avoid needless computations and by using the time saved to address important issues. In this way, perhaps we can make mathematics fun and satisfying and foster in our students a spirit of adventure and a thirst for knowledge.

Exploration in mathematics is not new. Mathematics began through observation. Early humans developed their mathematics through abstraction of natural forms and recognition of environmental patterns. Abstracting from specific instances to general properties and recognizing patterns still have an important place in mathematics today. Trial and error, guess and check, conjecture and proof are all a part of creating mathematics. An exploratory approach to teaching and learning recognizes this.

Modern psychology tells us that individuals learn through self-construction of ideas and interrelationships (see Bodner, 1986). Each person builds her or his own mathematics. The teacher acts as a guide during this building process. An environment of experimentation facilitates learning via construction, and calculators and computers can serve as tools for this mathematical experimentation. Exploration is in keeping with this empirical, constructivist view of mathematics teaching and learning.

Exploration, Problem Solving, and Thinking

Exploration as an approach to mathematics teaching and learning is closely related to two currently advocated educational themes: problem solving and higher-order thinking. In 1980, the National Council of Teachers of Mathematics proclaimed that "problem solving [should] be the focus of school mathematics in the 1980s" (NCTM, p. 1). Since

then, problem solving has become the focus of much mathematics education literature (see Suydam, 1987). Thinking, especially higher-order thinking, which many educators claim should be a major focus of the curriculum, has been a pervasive theme in recent general educational literature (see Patterson & Smith, 1986). The view of mathematics education as exploration captures the spirit and the intent of problem solving and higher-order thinking.

Exploration is related to problem solving.

The term exploration suggests the sustained investigation of mathematical concepts and interrelationships. Working toward the solution of a challenging problem can provide an environment for exploration, but problem solving and exploration are not synonymous. Exploration is often called for at various stages in the problem-solving process — understanding the problem, devising a plan, carrying out the plan, and looking back (see Polya, 1957) — and does embrace many of the commonly listed but infrequently taught problem-solving techniques: guess, check, and revise; work backwards; search for patterns; conjecture and try to prove or disprove; draw a graph; make a table; and solve a related problem. In other words, much of what we do when solving problems, especially complicated ones, is exploration. Problem solving, though intended to communicate a much deeper and much richer activity, is often interpreted as answer finding, and students end up working exercises rather than solving problems. To some, problem solving means doing story problems, but story problems too can become automatic, drill exercises, and many interesting problems are not based on stories.

Exploration is related to higher-order thinking.

Higher-order thinking may evoke an image of a human figure leaning forward with chin on fist; an image of contemplation without investigation. But, according to *Webster's Ninth New Collegiate Dictionary*, to explore means "to investigate, study, or analyze" (1986, p. 438). Patterson and Smith (1986) define higher-order thinking as occurring "when a person is engaged in active and sustained cognitive effort directed at solving a complex problem and when the person makes effective use of prior knowledge and experience in addressing the problem [italics omitted]" (p. 82). This definition conveys fairly well the notion of exploration used in this paper, but exploration has an added shade of meaning that is not emphasized by the definition. Exploration involves making sense of new, uncharted (for the student) territory. The explorer not only makes use of prior knowledge and experience but also constructs new knowledge and fits it into an existing structure. Exploration involves *learning*, not just thinking.

Exploration conveys more than either problem solving or higher-order thinking alone and is less likely to be misinterpreted. Perhaps this is why the word explore has such a prominent place in NCTM's *Curriculum and Evaluation Standards for School Mathematics:*

> To provide all students an opportunity to learn the mathematics they will need, the emphasis and topics of the present curriculum should be altered. More importantly, methods of instruction need to emphasize exploring, investigating, reasoning, and communicating on the part of all students . . . Classrooms should be places where interesting problems are explored using important mathematical ideas. For example, in various classrooms one could expect to see students recording measurements of real objects, collecting information and describing their properties using statistics, and exploring the properties of a function by examining its graph. (Commission on Standards for School Mathematics, 1987, p. 1)

This quote comes from a working draft of the *Standards*, but based on individual comments, regional meetings, and open discussions at the 1988 NCTM meeting in Chicago, there seems to be a consensus that this draft does provide an appropriate framework for the mathematics that should be taught in schools, and the final form of the *Standards* will likely be very similar. Exploration, as a major theme of the *Standards*, is linked to other themes: problem solving, reasoning, communication, and access to and appropriate use of technology.

How Can Exploration Be Taught?

Mathematics is sequential, and with few exceptions, there is a next course for which we always seem to be preparing our students. So the first step is to take a hard look at what content students really need in order to go on. We must ask and answer, What are the central concepts and critical supporting ideas and interrelationships? Douglas (1986) and Steen (1988) claim that, at least in calculus, there is some fat we can trim so that we can focus our students' attention on the meat of the subject. Calculators and computers speed computations and graphics, and thus both free time for the exploration of concepts and make new sorts of exploration possible. Once topical emphases and curricular themes are established, exploration can begin. The following list provides some starting points for creating an environment in which mathematical exploration can take place:

1. *Modeling Exploratory Behavior:* Instructors must exhibit the kind of approach to mathematics that they wish their students to follow. Planning, looking ahead, midstream checking, and looking back are all part of appropriate mathematical behavior. Pedersen and Polya (1984) give several excellent examples of problems that can be approached in many ways; students need to see their instructors investigating alternatives if they are to learn to do it themselves.

2. *Classroom Demonstrations:* Demonstrations can be memorable. Flipping graphs on overheads to demonstrate how to obtain the graph of an inverse, shining conical beams of light on planar walls and ceilings of darkened classrooms to illustrate conic sections, and drawing ellipses using a string taped to the chalkboard are but a few examples. The use of interactive software on a microcomputer linked to a large monitor or overhead projection palette creates a perfect set-up for class exploration.

3. *Classroom Discussions:* In order to generate a lively discussion, ask probing questions and seek out and test conjectures. Try to create a spirit of group exploration and cooperative learning. After exploring a concept using examples and nonexamples, get the class to generate a definition through a refinement process.

4. *Laboratory Activities:* Here again computers play a vital role. Students can explore individually with the instructor or lab assistant nearby. Laboratory assignments, whether within scheduled class time or as outside assignments, can be made progressively more challenging so as to build students' exploratory power. Assignments can also be individualized to provide challenge or remediation.

5. *Homework Problems:* Assign fewer problems and include some that require sustained effort. For example, you could assign a set of problems on series where you ask: Does each series converge or diverge? If it converges, can you prove it? How fast, or slow, does it converge? Explore the speed of convergence using a programmable calculator or a computer. If it diverges, in what sense does it diverge? It is interesting to explore on a hand-held or micro computer such series as $1/0! + 1/1! + 1/2! + 1/3! + \cdots$ and $1 - 1/3 + 1/5 - 1/7 + \cdots$ to see how dramatically different their rates of convergence are. The first

series converges to e very rapidly, and the second converges to $\pi/4$ very slowly. (See Simmons, 1985, pp. 715-716, 720-721, for further exploratory ideas relative to the second series.)

6. *Test Questions:* Again, less can be more. Ask fewer but more probing questions, such as, on a trigonometry test: Classify each equation as an identity, a conditional equation, or a null equation. If an identity, prove it. If conditional, solve it. If null, explain why. Such a group of items will sharpen students' understanding of trigonometric equations and the relationships among the various types.

Conclusion

This article advocates an exploratory approach to mathematics teaching and learning. Too often students view mathematics as a collection of facts and procedures to be memorized, and textbook exercises and even our own test questions frequently reinforce this shallow view. Overcrowded syllabi and the associated rapid-fire "covering" of the material do not help matters either. An exploratory approach requires extra preparation and effort on the part of the instructor, but pays off in an improved understanding and appreciation of mathematics on the part of the students. So let's encourage our students to explore and teach them how to explore as they do mathematics. Let's get them to realize that exploration is a natural and integral part of mathematics.

References

Bodner, G.M. (1986). Constructivism: A theory of knowledge. *Journal of Chemical Education, 63*, 873-878.

Commission on Standards for School Mathematics. (1987). *Curriculum and evaluation standards for school mathematics* (working draft). Reston, VA: National Council of Teachers of Mathematics. (Available from Thomas A. Romberg, Chair, NCTM Commission on Standards for School Mathematics, 1906 Association Drive, Reston, VA 22091)

Douglas, R.G. (Ed.). (1986). *Toward a lean and lively calculus: Report of the conference/workshop to develop curriculum and teaching methods for calculus at the college level* [MAA Notes No. 6]. Washington, DC: Mathematical Association of America.

Foley, G.D. (1987). Future shock: Hand-held computers. *The AMATYC Review, 9*(1), 53-57.

National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston, VA: Author.

Patterson, J.H., & Smith, M.S. (1986). The role of computers in higher-order thinking. In J.A. Culbertson & L.L. Cunningham (Eds.), *Microcomputers and education: 85th yearbook of the National Society for the Study of Education — Part I* (pp. 81-108). Chicago: University of Chicago Press.

Pedersen, J., & Polya, G. (1984). On problems with solutions attainable in more than one way. *College Mathematics Journal, 15*, 218-228.

Polya, G. (1950). Let us teach guessing. In *Etudes de philosophie des sciences, en hommage a Ferdinand Gonseth* (pp. 147-154). Neuchatel: Griffon.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method* (2nd ed.). Princeton, NJ: Princeton University Press.

Simmons, G.F. (1985). *Calculus with analytic geometry*. New York: McGraw-Hill.

Steen, L.A. (Ed.). (1988). *Calculus for a new century: A pump, not a filter* [MAA Notes No. 8]. Washington, DC: Mathematical Association of America.

Suydam, M.N. (1987). Indications from research on problem solving. In F.R. Curcio (Ed.), *Teaching and learning: A problem-solving focus* (pp. 99-114). Reston, VA: National Council of Teachers of Mathematics.

Webster's ninth new collegiate dictionary. (1986). Springfield, MA: Merriam-Webster.

More good reasons to use Munem and Tschirhart books

More problems
More examples
More on sets and set notation
More applications
More geometry
More pedagogical aids—self-tests, step-by-step procedures, ways to avoid common errors
More emphasis on important material
Less cost to your students

Munem and Tschirhart:

Beginning Algebra
Fourth Edition

Intermediate Algebra
Fourth Edition

Algebra for College Students
Second Edition

WORTH PUBLISHERS, INC.
33 Irving Place, New York, New York 10003 212-475-6000

Pandora's Rectangular Parallelipiped

Edited by $\quad 7988Z \times 11 \quad \int_{\theta}^{NER} SEC^2x\,dx$

In the first "Pandora's, etc." column, which appeared in the Fall, 1987, issue of the *Review*, a challenge was issued to readers to send in material of their own. One brave soul actually responded! (There are several possible explanations for this underwhelming response: The column had only one reader; most mathematicians don't have a sense of humor; most mathematicians *do* have a sense of humor. I won't take a poll.)

In any case, the challenge is hereby re-issued: submit some good stuff, or else. If anonymity is desired, you may mail your submissions in a plain brown envelope.

A former student of mine, Tom Rankin, gave me a long list of metric units, some of which follow. His source was a computer bulletin board to which he had access. Perhaps readers would care to submit others.

10 millipedes = 1 centipede;
10^{-6} phones = 1 microphone; 10^{6} phones = 1 megaphone;
10^{-9} goats = 1 nanogoat;
2×10^{3} mockingbird = 2 kilo mockingbird.

No limericks were forthcoming, but this one was found taped to my office door one year, just after midterm grades came out.

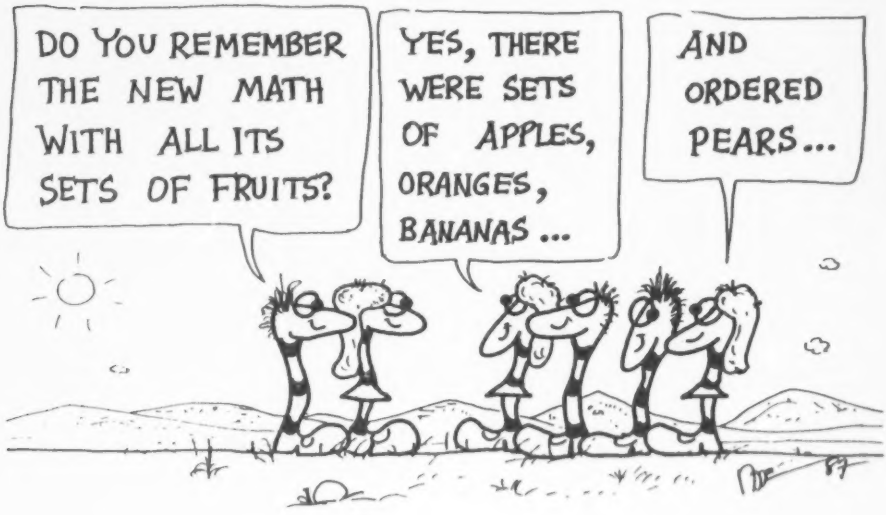
"What kind of a grader is he?",
They asked, about dear Dr. T.
Well, let us be terse —
Things *could* be worse.
He hasn't yet given a Z!

The Perfessor occupies an office just down the hall from mine. He has been there forever — legend has it that he set up his desk in the middle of an open field and they built the college around him. I am sure that he has a name, but "Perfessor" is all that he responds to.

The Perfessor is a number nut: he knows something out-of-the-ordinary about any number you can think of. Just the other night, for instance, at the monthly meeting of the ELM Club (to belong, one must correctly pronounce two of the three names Euler, L'Hospital, and Moebius), the treasurer announced that the club account had a balance of $1.43. Instantly, The Perfessor stated that any six-digit integer whose first three digits were identical to the second three was divisible by 143.

In later columns (should they exist), you will become better acquainted with The Perfessor. I would tell you more now, but he is unavailable for interviews. He has begun writing a book describing the interesting features of all numbers, and is he hung up in the middle of the unit interval.

Last but certainly not least is another "Footies" cartoon by Andrejs Dunkels.



Send contributions to this column, indicating source, if known, to:

Leslie R. Tanner, Editor
PRP Column, *The AMATYC REVIEW*
c/o The College of Idaho
2112 Cleveland Blvd.
Caldwell, ID 83605

MT 6. Identify the following quote:

"Logic is neither a science nor an act, but a dodge."

Become
A Better Teacher!

Put Your **Own** Study Aids In Your Students' Hands–
Lab Manuals, Study Guides, Class Notes.

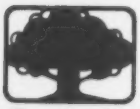
Choose Your Own Royalty: 10-20%
- For Yourself
- Your Department
- Your Favorite Scholorship

Send Us Your Copy:
- Camera Ready
- Or We Typeset

Quantity Printing:
- 100-10,000 Copies

If This Is Your Year To Publish, Let's Do Some
Business Together!

Write:



Tichenor Publishing
A Division Of T.I.S.
Box 669
Bloomington, IN 47402

Or Call: 1(812) 332-3307
1(800) 367-4002

Book Reviews

Edited by John J. Edgell, Jr.

COLLEGE ALGEBRA WITH APPLICATIONS, by L. Murphy Johnson and Arnold R. Steffensen. Scott, Foresman and Company; Glenview, Illinois, 1988. 483 pages ISBN 0-673-18354-8.

The basic topics of college algebra are discussed in an order used by most college algebra texts. The examples are splendidly detailed for easy comprehension by the student. There are several reasons why I like this textbook. First, examples state why they are being set up in a particular form and then state which property or procedure is being used in each step of the problem. Second, cautions are inserted at strategic times to warn students about common mistakes that are often made. A typical comment we hear from students is "Why are we studying this? What can this really be used for in the real world?" The text answers these questions for the student. Each chapter is preceded by one or two examples of problems to be discussed in the chapter with their application to the "real" world. This is also done with the problem sets for each section. These applications involve many areas, such as business, engineering, construction, manufacturing, aeronautics, investment, and consumer applications, just to name a few.

I found the book to be very well written and very student-reader-oriented. The examples are detailed so well that every student should be able to follow them, especially since each step documents what is being done. It is one of the best college algebra texts I have seen and would be the ideal book to use if a student were taking the course by correspondence. I wish that my students were using this book this semester.

Reviewed by Paul Foutz, Southwest Texas State University, San Marcos, Texas.

ESSENTIAL OS/2 FOR NEW OS/2 USERS, by Judd Robbins, 1988 Sybex Inc., Almeda, CA 94501. 347 pages. ISBN 87-630-45.

This book is designed for experienced DOS users who have been looking for material relating information on IBM's new multitasking operating system, OS/2. For these people, this book does not disappoint. However, those of us prepared to read a book that will clearly demonstrate how OS/2 works, as well as how one uses OS/2, should choose a different book. The author of this book provides too little information pertaining to the manner in which OS/2 works, (which would be of most interest to software developers), and not enough information on how to use OS/2, (which would be of most interest to every future OS/2 user).

The experienced DOS user will find this book's ten chapters organized fairly well, with major emphasis placed on OS/2 commands. The author does cover "batch" files, as well as the "config. sys" file and its associated options, and IBM's text editor, EDLIN.

Chapter five consists of a comprehensive reference for all OS/2 commands. Thus, if the experienced DOS user needs a quick reference guide to those DOS commands that can also be used in OS/2, as well as commands designed specifically for OS/2, this chapter may be worth the price of the book. However, we must realize that references do not offer the same in depth presentations as tradition writing styles, and as such, are not read the same way. One usually accesses a reference of operating system commands, when prior knowledge of the commands found within the reference exist, and a brief refresher is needed.

In conclusion, this book cannot be used as a textbook for an operating systems course, or a course in OS/2. I bought it because I wanted to read everything relating to OS/2 that I could find. I'm not so sure you should do the same.

Reviewed by Professor Louis V. Gioia, Department of Mathematics/Statistics Computer Processing, Nassau Community College, Garden City, NY 11530.

OS/2 PROGRAMMING GUIDE, Ed Iacobucci. Osbourne McGraw Hill, Berkeley, California, 1988, 1100 pages. ISBN 0-07-881300-X.

It is certain that DOS users will soon be learning how to use IBM's new microcomputer operating system, OS/2. It is also certain that a profusion of text books teaching OS/2 will be available. Let me make one important recommendation. Whether you want to learn OS/2 for your own personal enrichment, or for the intention of eventually teaching it in the classroom, you must first read this book.

This book is not a classroom text. I would certainly not choose it as a primary text for a course teaching OS/2. It's not designed for that purpose. The book does provide a solid description of how DOS works, followed by the limitations of DOS. These limitations lead the reader into a presentation on the need for a better microcomputer operating system, OS/2. Once the reader understands the need for OS/2, over 1000 pages are devoted to helping the reader understand how OS/2 works with a microprocessor, to shoulder user requests.

The reader will find the discussions informative, fairly complex, and geared towards software developers. These facts should not prevent you from reading it. I cannot imagine anyone attempting to teach OS/2 without first reading this book. I think that this book must be added to your library.

The authors coverage of OS/2's memory management, and file management facilities, as well as discussions on how OS/2 is capable of harnessing the power of the 80286 and 80386 microprocessor, are superb. These sections represent the most fascinating parts of the book. Certainly, these sections should be read carefully. In fact, students of operating systems courses would benefit greatly by reading these sections because OS/2 is basically a microcomputer operating system whose capabilities are derived from those of main frame computer systems.

Finally, the reader is immediately aware of the fact that the author leads the OS/2 design team. This tells us all two things; the book is highly technical, and very thorough. Don't expect to read this once and assimilate all that is within it. Take your time, and enjoy Iacobucci's labor of love.

Reviewed by Professor Louis V. Gioia. Department of Mathematics/Statistics Computer Processing, Nassau Community College, Garden City, NY 11530.

Send reviews to: Dr. John J. Edgell, Jr.
Editor, Book Reviews
The AMATYC Review
Mathematics Department
Southwest Texas State University
San Marcos, Texas 78666

The Prindle, Weber & Schmidt Series in Mathematics Presents the Masters

***NEW FOR 1989**

EARL W. SWOKOWSKI
Marquette University

***FUNDAMENTALS OF COLLEGE ALGEBRA, Seventh Edition**

***FUNDAMENTALS OF TRIGONOMETRY, Seventh Edition**

***FUNDAMENTALS OF ALGEBRA AND TRIGONOMETRY, Seventh Edition**

***ALGEBRA AND TRIGONOMETRY WITH ANALYTIC GEOMETRY, Seventh Edition**

PRECALCULUS: FUNCTIONS AND GRAPHS, Fifth Edition

CALCULUS WITH ANALYTIC GEOMETRY, Second Alternate Edition

CALCULUS WITH ANALYTIC GEOMETRY, Fourth Edition

DENNIS G. ZILL
Loyola Marymount University

***A FIRST COURSE IN DIFFERENTIAL EQUATIONS WITH APPLICATIONS, Fourth Edition**

***DIFFERENTIAL EQUATIONS WITH BOUNDARY-VALUE PROBLEMS, Second Edition**

CALCULUS WITH ANALYTIC GEOMETRY, Second Edition

PWS-KENT Publishing Company
20 Park Plaza
Boston, Ma. 02116
A Division of Wadsworth, Inc.
1-800-343-2204 (in Mass. 617-542-3377)

WADSWORTH, INC.-- WINNER OF THE NATIONAL ASSOCIATION OF COLLEGE STORES
"Publisher of the Year"
Award for 1988!

The Problem Section

Edited by Michael W. Ecker
Mathematics Department
Pennsylvania State University
Wilkes-Barre Campus
Lehman, PA 18627

Welcome to still another Problem Section!

Continued thanks to those of you sending in problem solutions. I still could use some problem proposals, however, even though I do have a reserve ready.

As I reported recently, I am even more pleased that participation is picking up and perking up. Indeed, last issue I cautiously stated that the deadline for set N problems could be extended, yet I began receiving solutions for set N within a month after getting them for set M. This, of course, is due to the timetable of publication which somehow made most of us get our last two issues of the *AMATYC Review* within a month of each other.

Speaking of the latest *Review*, compliments to all concerned with overall editing and composition. As AMATYC grows, this organization and its journal seem to get better and better, making me proud to be even just a tiny cog in the overall machinery.

On to "business" with this reminder: The *AMATYC Review* Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics, but particularly those which have applicability of some kind in the lives of two-year college math faculty and students. Please eschew routine and familiar problems, however. If you're not sure, send something in anyway. At the least, we may be able to modify it to turn a routine question into a real think piece; at best, you may have a true gem that will bring you instant fame among your peers. Send your problems and solutions, preferably typed, plus generalizations, comments, special cases, observations, and improvements to:

Michael W. Ecker
Problem Section Editor
The AMATYC Review
129 Carol Drive
Clarks Summit, PA 18411

Please send me a mailing label (or self-addressed card or envelope) bearing your courtesy title, name, and address for acknowledgement of solutions or proposals.

Lastly, please note the error in one example in the statement of problem M-5 two issues ago, the one about $1/a + 1/b = .(a+b)$ being sometimes true. I had "$1/3 + 1/7 = .21$" as an illustration, when I intended the right side to be .10 instead. Thanks to several eagle-eyed reader-problemists for the alert.

Set 0 Problems. Due for ordinary consideration March 1989. Regardless of deadline, no problem is ever closed permanently, and new insights are always welcome. I believe most readers will find the majority of the problems now on the easy side, but bear with me as I fine-tune. Besides, some of these are just too nice not to include.

Problem O-1. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus.

For a right triangle with legs a and b and hypotenuse c, the inequality

$$a,b < c < a+b$$

reflects elemental truths of Euclidean geometry. Suppose the percentage increase in the

sum of t[illegible] compared to the hypotense matches the percentage increase in the hypoten[illegible]pared to one of the legs. What can we conclude?

Problem O-2. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus.

Prove that all right triangles with sides in arithmetic progression are similar to each other.

Comment: I originally used this this past year in the PSMATYC Newsletter, the publication of the Pennsylvania state affiliate of AMATYC, for which I serve similar duty as problem box editor. My apologies to my Pennsylvania colleagues for the repetition.

Problem O-3. Proposed by Howard K. Hilton, Chicago, IL.

Let b, c, d and f, g, h be integral lengths of sides of two different triangles, each with a 120-degree angle. Verify that bf is the length of the long side of two new 120-degree triangles, one similar to each of the original two.

After verifying this, prove that bf is the long side of two other 120-degree triangles whose sides are integers. (This result is analogous to what well-known theorem concerning Pythagorean triangles?)

Problem O-4. Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, MA.

It should be painfully obvious that no finite collection of straight lines can cover all the lattice points in the plane. (A lattice point is a point with integral coordinates.)

Prove it.

Problem O-5. Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, MA.

Prove that 31^2 divides $(5^n + 279n - 1)$ if n is an integral multiple of 3.

Problem O-6. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus.

An astronaut with perfect vision is 10 miles up, as measured above a perfectly spherical earth, on a day with unlimited visibility. Consider these questions for this particular moment (solve one or more of these parts):

a) How far away is she from the most distant point she can make out on the horizon?
b) How far apart, at most, are any pair of such most distant points from each other as measured by a path along the earth?
c) How must surface area of the earth is visible to the astronaut?

Problem O-7. Contributed by Troy E. O'Brien (graduate student), Central Michigan University, Mt. Pleasant, MI, paraphrased as follows:

Given two concentric circles, what is the area of the annular region between them if a chord of the outer circle is tangent to the inner circle and said chord has a length of 100?

Problem Editor's Comment: Troy reported finding this in Martin Gardner's book, *Aha! Insight*. I offer this follow-up question: Suppose that the radii are integers. What are the possible pairs of values for the radii then?

Problem O-8. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus.

A unitary divisor d of a positive integer n is a positive integer divisor of n which is relatively prime to $d' = n/d$. For instance, the unitary divisors of 50 are 1, 2, 25, and 50, whereas the non-unitary divisors are 5 and 10. Let w(n) equal the number of unitary

divisors of n. The example here shows w(50) = 4. Note that w(6) = 4 as well, showing w is far from being one-to-one.

a) Write a formula for w(n), the number of unitary divisors of a number n.

b) What is the range of this function w? Why?

Problem O-9.* Proposed by Samuel Yates, Delray Beach, FL.

Any natural number $n>2$ is a repunit, a number representable as a string of ones in at least one base b, viz: base $b = n-1$. For instance, 7 is 11_6 (11 in base 6) as well as 111_2 (111 in base 2).

There are at least two numbers representable in three different bases: 31 is a repunit in bases 2, 5, and 30; 8191 is one in bases 2, 90, and 8190.

Are there other numbers that are representable as repunits in three or more bases? Can anybody shed any light in the form of an existence proof, computer search, brute force, or some combination?

Problem Editor's Comment: Samuel Yates — "Mr. Repunits" — is an authority on this topic. His book *Repunits and Repetends* is a valuable source of information, and it is available directly from Mr. Yates for roughly $10 plus shipping. His address is 157 Capri-D, Kings Point, Delray Beach, FL 33484.

The asterisk on the problem indicates that the proposer did not include a solution, and the problem editor, offhand, doesn't have one either. However, the case of having two representations is easy. Simply choose numbers n of the form $b^2 + b + 1$ and then n will have the form 111 in base b as well as 11 in base n−1.

Problem O-10. Proposed by Stephen Plett, Fullerton College, Fullerton, CA.

Choose k lattice points in the plane such that no three are collinear, yet the distance between any two is an integer. Can you do this for k=6? 8? 10? What about arbitrary k?

I do have more problems on hand, but no more room. Because of space limitations, solutions to set M problems will be deferred until next issue, along with at least some of the set N ones. Here are the others owed.

Solutions

Perfect Squares from Linear Recurrence

K-1. Proposed by Stanley Rabinowitz.

Find a linear recurrence with initial conditions which generates precisely the sequence of perfect squares. (What about perfect cubes, etc.?)

Solutions were submitted by Glen E. Mills (Colonial Senior High, Orlando, FL), Steven Blasberg (West Valley College, Saratoga, CA), and the proposer. (Solutions have been edited to correct minor technical errors.)

For squares it's $x_n = 2x_{n-1} - x_{n-2} + 2$ for $n\geq 2$, with initial conditions $x_0=0$ and $x_1=1$. There are several ways to see this using second differences, patterns, undetermined coefficients, or some combination of these. Perhaps the simplest heuristic to understand the foregoing is to start with the identity $a^2 = 2(a-1)^2 - (a-2)^2 + 2$, noting the binomial coefficients, to solve the original problem, as Mills did. Can you take it from here, readers?

Glen Mills argued essentially that, if we note that the right member's coefficients are binomial coefficients and the nonhomogeneous term 2 is a factorial, we can generalize to obtain results for not just cubes, but all powers. For fixed m, we get:

$$(m^{th} \text{ powers})\ n^m = x_n = \left\{ \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} x_{n-i} \right\} + m!\ (n\geq m);\ x_i = i^m\ (i=0,1,\ldots,m-1).$$

Steve Blasberg, on the other hand, recognized that there are homogeneous as well as nonhomogeneous solutions. Besides the foregoing, he had the homogeneous solution:

$$(m^{th} \text{ powers}) \quad n^m = x_n = \sum_{i=1}^{m+1} (-1)^{i-1} \binom{m+1}{i} x_{n-i} \quad (n \geq m+1); \; x_i = i^m \; (i=0,1,\ldots,m).$$

Note that there is a tradeoff; the homogeneous solution requires one extra initial condition and m^{th} powers are in terms of $(m+1)$ previous terms instead of m terms.

$$a^b = c$$

K-2. This problem about rational and irrational values of a,b,c, besides being mentioned previously, has been discussed in *AMATYC Review* articles. See Vol. 4, No. 1, Fall/Winter 1982, pages 35-36. Steve Blasberg sent in a solution, too.

Telescoping Series

L-2. Proposed by Bjorn Poonen (student), Harvard College.

Evaluate the following convergent series:

$$\sum_{n=1}^{\infty} \frac{[\sqrt{4n+1}] - [\sqrt{4n}]}{n}$$, where the brackets indicate the greatest-integer function.

Similar solutions were submitted by Lawrence G. Gilligan (University of Cincinnati, OH), Ann E. Watkins (Los Angeles Pierce College, Woodland Hills, CA), Harris Shultz (California State University-Fullerton, CA), Stephen Plett (Fullerton College, CA), and the proposer. A composite of their solutions follows:

The numerator is zero unless $4n+1$ is a perfect square, $(2k+1)^2$, in which case numerator equals 1. Substitution and simplification yield the following:

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1.$$

Iterated Function

L-3. Proposed by Bjorn Poonen (student), Harvard College.

Let $f_0(x) = \sin \pi x$, and $f_{n+1}(x) = f_0(f_n(x))$ for $n \geq 0$. Find the number of solutions in $[0,1]$ to $f_n(x) = 1/2$.

Solution with generalization by Ann E. Watkins, Los Angeles Pierce College, Woodland Hills, CA:

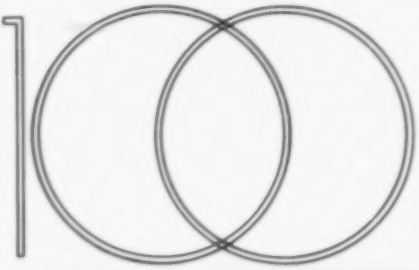
There are two distinct solutions on $[0,1]$ to $f_0(x) = \sin x = k$ for all k in $[0,1]$: $(\sin^{-1} k)/\pi$ and $1 - (\sin^{-1} k)/\pi$. (These terms must be distinct, as $k<1$.)

Assume there are 2^n distinct solutions to $f_{n-1}(x) = k$. Now, $f_n(x) = k = \sin(\pi f_{n-1}(x))$ has solutions whenever $f_{n-1}(x)^{n-1}$ equals one of the terms above. By hypothesis, since these terms are constants, each of these contributes 2^n solutions. Since the terms are distinct, all $2 \cdot 2^n = 2^{n+1}$ solutions are distinct.

This completes the induction that shows that there are 2^{n+1} solutions, in general, to $f_n(x) = k$ for any k satisfying $0 \leq k < 1$. This includes the special case of $k = 1/2$ in the original problem statement.

Comparable solutions were also submitted by Stephen Plett (Fullerton College, CA) and the proposer.

That's all for now. Until next time, may all your problems be purely mathematical!



MATHEMATICS PROGRAMS FOR THE IBM® PC

For IBM-PC or IBM PS/2 (DOS 2.0 or later) and all compatibles. Programs are written in easy-to-read BASIC with ample REM statements to facilitate program modification. You may adapt these programs to your own uses and copy them onto other disks. Topical headings and the number of programs per heading are as follows:

Patterns Among Numbers (21)
Numbers Without Discernible Patterns (6)
Functions and Curves (10)
Roots and Complex Numbers (12)
Lines (5)
Conics (7)
Polynomials (6)
Super Arithmetic (8)
Matrix Algebra (14)
Topics in Calculus (6)
Miscellaneous (5)

The sixty page manual that comes with the disk provides concise sections on ***Topics and Notes, First Run Suggestions, Suggested Program Modifications, and References.***

Institutional, library, and individual orders $70.00 plus 10% postage and handling. Be sure to specify disk size (5¼ or 3½). Send purchase order or check to:

Mathematics Software Co.
419 S. Boeke Road
Evansville, Indiana 47714

Software Reviews

Edited by Shao Mah
Red Deer College
Red Deer, AB, Canada T4N 5H5

Title: INTERACTIVE LESSONS I: ARITHMETIC AND ALGEBRA
INTERACTIVE LESSONS II: ALGEBRA

Distributed By: H & H Publishing Company, 1117 Webb Drive, Clearwater, FL 33515

Price: $318.00 U.S. / set of 7 diskettes (see "Content Description")

Minimum Hardware Requirements: Apple IIe computer, single disk drive

Content Description: INTERACTIVE LESSONS I and II are comprised of 14 stand-alone diskettes. The topics are:

Lesson I

1. Least Common Multiples and Highest Common Factors
2. Percent and Solving Percent Problems
3. Evaluating Numerical Expressions
4. Simplifying Algebraic Expressions
5. Solving Equations with Counting Numbers
6. The Arithmetic of Integers (Signed Numbers)
7. Solving Equations with Integers

Lesson II

1. Solving Equations with Rational Numbers
2. The xy-Plane
3. Graphing Two-Variable Equations
4. Solving Pairs of Simultaneous Equations
5. Using Counting Numbers as Exponents
6. Using Integers as Exponents
7. Operations on Polynomials

Each diskette is further subdivided into independent, discrete units which are menu-driven. As there is no management system for this program, a user can adjust his interaction according to the amount of time available and individual skill level.

The topics covered are those which students commonly have difficulty with. Pedagogical treatment of each topic is standard. Each objective is presented by a short statement and/or example. The student is then given the opportunity to respond to a related question. Feedback is immediate. For a correct response, positive reinforcement precedes the presentation of the next objective. When a response is incorrect, a second chance is given. After a second incorrect response, the program gives the correct answer and an explanation. The user is then directed to either continue with the lesson or seek other help, according to the number of errors.

Documentation consists of two laminated sheets which describe the general content of the programs, how to start up the diskettes, disk topics, and presentational format.

Content Evaluation: The treatment of each topic is standard. The explanations are acceptable and the use of graphics is effective. The amount of user interaction is good and consists of more than simply page-turning.

There are, however, enough inadequacies to question the value of investing in this software. Some of these are:

- The positive reinforcement is overdone to the point of seeming ridiculous.
- Music is used to indicate that a student has been referred for extra help. The music is juvenile and may be embarassing for the user.
- The interruptions are long and frequent when the program has to "think".
- It is time-consuming to move from one unit to another.
- There are too few problems and these tend to have the same format, thus problem-solving is reduced to patterning rather than analyzing.
- Answer formats were not specified. This created complications when student responses were correct but were input in a non-compatible format.

Summary: My hunch is that there are other, less expensive programs on the market which cover the same topics in a manner more satisfying to the user.
Reviewed by Rosalind Wild, Learning Assistance Centre, Lakeland College, Vermilion, AB, Canada

---

Mathematical Trivia (MT)

by Ralph Selig

TRIVIA ANSWERS:

MT 1. **2047** and **8191**

$2^2 - 1 = 3$
$2^3 - 1 = 7$
$2^5 - 1 = 31$
$2^7 - 1 = 127$
$2^{11} - 1 = 2047$

Note: Take 2 and raise it by consecutive primes; then subtract 1.

MT 2.

1. Draw a straight line from any point to any point.
2. Produce a finite straight line continuously in a straight line.
3. Describe a circle with any center and any distance.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

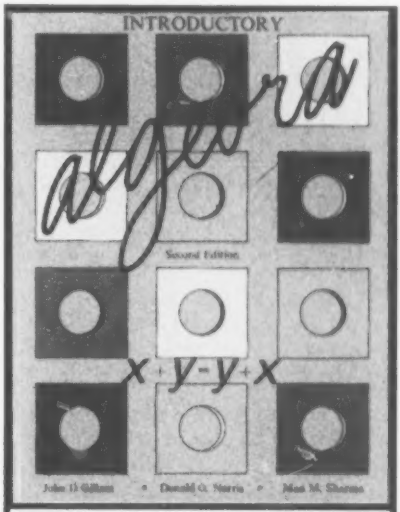
MT 3. Jean Bernoulli

MT 4. G. W. Freiherr von Leibniz

MT 5. Henry Briggs

MT 6. Benjamin Jowett

Questions appear throughout journal. Look for them.



Introductory Algebra
second edition
John D. Gillam, Donald O. Norris and Man M. Sharma
1986/384 pages/Paper
ISBN 0-8403-3915-1

Sharpen your students' algebra skills.

Introductory Algebra is more than just another algebra text; it goes beyond the basics to make algebra easy.

Each concept is presented in an easy four-step process by:

1. Using examples to illustrate new concepts
2. Abstracting formal rules, procedures, or definitions from the examples
3. Further illustrating rules, procedures, or definitions with more examples
4. Reinforcing concepts by supplying a large number of problems to be solved

Introductory Algebra includes many other outstanding features, including:

- Several illustrative examples with explanatory comments.
- An arrow sign focusing attention on notes of caution or important results.
- Important rules and methods boxed for easy reference and review.
- Chapter summaries at the end of each chapter.
- Answers to all problems at the back of the book.

Computer software will ease your teaching by helping you manage your course, including:

- A question bank with over 2000 items.
- The ability to generate a number of tests.
- Provisions for taking practice tests and make-up tests at the computer.
- Test grading, record keeping and progress reports for each student.

I would like a complimentary copy of **Introductory Algebra** to review for adoption consideration.

Name _____ State/ZIP _____
Affiliation _____ Phone _____
Address _____
City _____
04781746 A06 bk#3915

Kendall/Hunt Publishing Company
2460 Kerper Boulevard PO Box 539
Dubuque, Iowa 52001

AMATYC Reviewers

EDITORIAL PANEL

Agnes Azzolino	Middlesex County College	Edison, NJ
Ione Boodt	Univ. of Indianapolis	Indianapolis, IN
Mike Davidson	Modesto J.C.	Modesto, CA
Carla Thompson	Tulsa Junior College	Tulsa, OK
Brian Hickey	East Central College	Union, MO
Gordon W. Hoagland	Ricks College	Rexburg, ID
Nelson G. Rich	Nazareth College	Rochester, NY
Martha M. Wood	Clayton State College	Morrow, GA

REVIEWER PANEL

Darrell H. Abney	Maysville C.C.	Maysville, KY
Paul E. Bland	Eastern Kentucky University	Richmond, KY
Wilfred E. Boykin	Univ. Central Ct. State	New Britain, CT
Randall Brian	Vincennes University	Vincennes, IN
Pansy Brunson	Western Kentucky University	Bowling Green, KY
Alan Chutsky	Queensborough C.C.	Bayside, NY
Patricia Costello	Eastern Kentucky University	Richmond, KY
John DeCoursey	Vincennes University	Vincennes, IN
George Dorner	Harper College	Palatine, IL
David Dyer	Prince Georges	Largo, MD
Joseph R. Fieldler	Ohio State University	Columbus, OH
Gregory D. Foley	Ohio State University	Columbus, OH
Charles Franke	Eastern Kentucky University	Richmond, KY
Barbara R. Gab	Prince Georges	Largo, MD
Anne Grams	Embry Riddle Aeronautical Univ.	Daytona Beach, FL
Robert J. Greger	Harford C.C.	Bel Air, MD
Ronald J. Harkins	Miami University	Hamilton, OH
Peter Herron	Suffolk C.C.C.	Selden, NY
Calvin J. Holt	Paul D. Camp C.C.	Franklin, VA
James T. Johnson	Modesto Jr. College	Modesto, CA
Sharon Kunoff	C.W. Post Campus — LIU	Greenvale, NY
Edward Laughbaum	Columbus State C.C.	Columbus, OH
Deborah Levine	Nassau C.C.	Garden City, NY
Ved P. Madan	Red Deer College	Alberta, CANADA
Dominic Magno	Harper College	Palatine, IL
Pamela E. Matthews	Willamette University	Salem, OR
Mary McCarty	Sullivan C.C.C.	Loch Sheldrake, NY
Terry A. Nyman	UW-Fox Valley	Menasha, WI
Carol Olmstead	Univ. of Nevada	Reno, NV
Loren W. Pixley	Richland C.C.	Decatur, IL
Dennis Reissig	Suffolk C.C.C.	Selden, NY
Jack W. Rotman	Lansing C.C.	Lansing, MI
Karen J. Schroeder	Bentley College	Waltham, MA
Ralph Selig	Fairleigh Dickinson University	Hakensack, NJ
Lowell Stultz	Kalamazoo Valley C.C.	Kalamazoo, MI
Leslie R. Tanner	The College of Idaho	Caldwell, ID
Travis Thompson	Harding University	Searcy, AR
August J. Zarcone	College of DuPage	Glen Ellyn, IL

The Mathematical Symbol for Quality

Brand-new in 1989
BASIC MATHEMATICAL SKILLS, 2/e
BEGINNING ALGEBRA, 2/e
INTERMEDIATE ALGEBRA, 1/e
James Streeter, Gerald Alexander and Donald Hutchison, all of Clackamas Community College

The Streeter Series is available with a remarkably complete package of supplements including: Instructor's Manuals, Student Solutions Manuals, Test Banks (in computerized and booklet forms), Annotated Teacher's Editions, software, and videos.
The Streeter Series: On the road to success with your students

Brand-new in 1989
FINITE MATHEMATICS, 3/e
Daniel Maki and Maynard Thompson, both of Indiana University

CALCULUS FOR BUSINESS, ECONOMICS, SOCIAL AND LIFE SCIENCES, 4/e
Laurence D. Hoffmann and Gerald L. Bradley of Claremont McKenna College

Brand-new in 1989
COLLEGE ALGEBRA, 4/e
COLLEGE ALGEBRA WITH TRIGONOMETRY, 4/e
PRECALCULUS: FUNCTIONS AND GRAPHS, 2/e
Raymond A. Barnett, Merritt College and **Michael Ziegler,** MarquetteUniversity

The Barnett-Ziegler Series comes complete with an extensive line of supplements that include: Instructor's Manuals, Student Solutions Manuals, Overhead Transparencies, Test Banks (in computerized and booklet forms), and software.
It's coming this fall– don't miss it!

And proven winners from 1987/88
CALCULUS AND ANALYTIC GEOMETRY, 4/e **Sherman K. Stein,** University of California at Davis

PLANE TRIGONOMETRY, 6/e
E. Richard Heineman, Texas Tech University

ELEMENTARY LINEAR ALGEBRA
William L. Perry, Texas A&M University

APPLIED MATHEMATICS FOR BUSINESS, ECONOMICS, AND THE SOCIAL SCIENCES, 3/e
Frank S. Budnick, University of Rhode Island

To order your examination copy, please write: McGraw-Hill Book Company, College Division, P.O. Box 446, Highstown, N.J. 08520

The Mathematical Symbol for Quality

AMATYC Institutional Members

Addison-Wesley Publishing College	Reading, MA	01867
Alexandria Technical Institute	Alexandria, MN	56308
American River College	Sacramento, CA	95841
Anchorage Community College	Anchorage, AK	99504
Anoka Ramsey Community College	Coon Rapids, MN	55433
Aquinas Jr. College	Nashville, TN	37205
Ball State University	Muncie, IN	47306
Brooks-Cole Publishing Company	Pacific Grove, CA	93950
Bucks County Community College	Newton, PA	18940
Burlington County College	Pemberton, NJ	08068
Butte College	Oroville, CA	95965
C. S. Mott Community College	Flint, MI	48503
Casper College	Casper, WY	82601
Chabot College	Hayward, CA	94545
Charles Co. Community College	La Plata, MD	20646
Chemeketa Community College	Salem, OR	97309
Clackamas Community College	Oregon City, OR	97045
Clark Technical College	Springfield, OH	45501
Cochise College	Sierra Vista, AZ	85635
Columbus Technical Institute	Columbus, OH	43215
D. C. Heath and Company	Lexington, MA	02173
Dawson College	Montreal, QUE (Canada)	H4C 2R8
DeKalb Community College	Decatur, GA	30034
Delta College	University Center, MI	48710
Dodge City Community College	Dodge City, KS	67801
East Central College	Union, MO	63084
EN Boekhandel B V	1000 GC Amster., Hol., Netherland	
Florida Jr. College-Jacksonville	Jacksonville, FL	32205
Front Range Community College	Westminister, CO	80030
Fullerton College	Fullerton, CA	92634
Gateway Technical Institute-Racine	Racine, WI	53402
George Brown College	Toronto, Ontario, (Canada)	M5T 2T9
H & H Publishing Company, Inc.	Clearwater, FL	33575
Harford Community College	Bel Air, MD	21014
Harper College	Palatine, IL	60067
Henry Ford Community College	Dearborn, MI	48128
Houghton Mifflin Company	Boston, MA	02107
Howard Community College	Columbia, MD	21044
Idaho State University	Pocatello, ID	83201
Illinois State University	Normal, IL	61761
Johnson County Community College	Overland Park, KS	66210
John Wiley and Sons	New York, NY	10158
Kansas Technical Institute	Salina, KS	67401
Kean College of New Jersey	Union, NJ	07083
Kennedy-King College	Chicago, IL	60621
Lake City Community College	Lake City, FL	32055
Lane Comm. College Dept. of Math	Eugene, OR	97405
Lexington Community College	Lexington, KY	40506
Madison Area Technical College	Madison, WI	53704
Mercer County Community College	Trenton, NJ	08690
Metropolitan Technical Comm. College	Omaha, NE	68103
Middlesex County College	Edison, NJ	08818
Montgomery College	Takoma Park, MD	20912
Moraine Valley Community College	Palos Hills, IL	60465
MT Hood Community College	Gresham, OR	97080
MT Royald College Lincoln Park Campus	Calgary, Alb (Canada)	T3E 6K6
New Mexico Military Institution	Roswell, NM	88201
New York City Technical College	Brooklyn, NY	11201
Normandale College Learning Resources Ctr.	Bloomington, MN	55431
North Hennepin Community College	Minneapolis, MN	55455
North Idaho College	Coeur D'Alene, ID	83814
Northern Arizona University	Flagstaff, AZ	86011
Oakton Community College	Des Plaines, IL	60016
Olympic College	Bremerton, WA	98310
Parkland College	Champaign, IL	61821
Penn State University-Capital Campus	Middletown, PA	17057
Pierce Jr. College	Philadelphia, PA	19102
Polk Community College	Winter Haven, FL	33881
Portland Community College	Portland, OR	97219
Prentice-Hall Inc.	Englewood Cliffs, NJ	07632
Prince George's Community College	Largo, MD	20772

BEST COPY AVAILABLE

Raritan Valley Community College	Somerville, NJ	08876
Raymond Walters College Library	Cincinnati, OH	45236
Richland Community College	Decatur, IL	62526
Ricks College	Rexburg, ID	83440
San Francisco State University	San Francisco, CA	94132
Santa Monica College	Santa Monica, CA	90405
Santa Rosa Jr. College	Santa Rosa, CA	95401
Saunders College Publishing Company	Philadelphia, PA	19105
Schoolcraft College	Livonia, MI	48152
Scott Foresman Company	Glenview, IL	60025
South Central Community College	New Haven, CT	06511
Southern Alberta Institution of Tech.	Calgary, Alb (Canada)	T2M OL4
Southeast Virginia Community College	Richlands, VA	24641
St. Augustine College	Chicago, IL	60640
St. Bernard Parrish Community College	Chalmette, LA	70043
SUNY Ag & Tech College	Alfred, NY	14802
Tarrant County Jr. College	Ft. Worth, TX	76119
Three Rivers Community College	Poplar Bluff, MO	63901
University of Wyoming	Laramie, WY	82071
Volunteer State Community College	Gallatin, TN	37066
Wallace State Community College	Selma, AL	36702
Waterbury State Technical College	Waterbury, CT	06708
West Publishing Company	Moss Beach, CA	94038
Westark Community College	Ft. Smith, AR	72913

INSTITUTIONAL MEMBERSHIP

Colleges, publishers, and other institutions can support AMATYC by being Institutional Members. As indicated below, the annual membership fee is $75. Subscriptions to *The AMATYC Review* and *The AMATYC News* are included in this fee. Of particular importance to collegiate institutional members will be the new AMATYC committee for mathematics department chairpersons.

An additional benefit of an institutional membership is one *complimentary* AMATYC convention registration. Future conventions, which are held in the Fall, will be in Calgary (1988) and Baltimore (1989). Institutional members support the excellence of the programs at these annual conventions.

INSTITUTIONAL MEMBERSHIP APPLICATION

The American Mathematical Association of Two-Year Colleges

Mail to:
Karen Sharp, AMATYC Treasurer, Mott Community College, Flint, Michigan 48502

AMATYC COLLEGE CONTACT PERSON _____ POSITION _____

NAME OF INSTITUTION _____

ADDRESS _____

CITY _____ STATE _____ ZIP _____

Membership Fee — $75 in U.S. funds payable to AMATYC (includes *The AMATYC Review*, *The AMATYC News*, and one complimentary convention registration)

LIBRARY SUBSCRIPTION

Annual Library Subscription Fee: $15 [Includes *The AMATYC Review* (Fall and Spring) and *The AMATYC News* (Fall, Winter and Spring)]

Name of Library

Name of Institution

Address

City State Zip

NOTE: This application form is your invoice for the subscription fee and payment must accompany the completed application. Retain a photocopy of this form for your records if necessary. Please make check payable in U.S. funds to AMATYC and mail to:

Karen Sharp, AMATYC Treasurer, Mott Community College, 1401 E. Court Street, Flint, Michigan 48502

Future AMATYC National Conventions

CALGARY — 1988
October 27-30, 1988
Palliser Hotel
Co-Chairs: Steven S. Terry
Ricks College
Rexburg, ID 83440

Shao Mah
Red Deer College
Red Deer, Alberta
Canada T4N 5H5

BALTIMORE — 1989
October 25-29, 1989
Hyatt Regency
Chair: Barbara Gale
Prince George's CC
Largo, MD 20772

DALLAS — 1990
October 17-21, 1990
Fairmount Hotel
Co-Chairs: Eddie Robinson
Cedar Valley College
3030 N. Dallas Ave.
Lancaster, TX 75134

Tommy Thompson
Brookhaven College
3939 Valley View Lane
Farmers Branch, TX 75234

SEATTLE — 1991

INDIANAPOLIS — 1992

89

WHY JOIN AMATYC?

The American Mathematical Association of Two-Year Colleges (AMATYC) was established in 1974 to provide a unique, national forum for two-year college mathematics educators. Today, AMATYC is the only national organization that exclusively serves the needs and purposes of this group.

AMATYC holds a national convention annually in a major city. AMATYC encourages two-year college mathematicians to assume responsible leadership positions; to interact on an equal basis with four-year university personnel; to be members on a proportional basis of national steering or policy committees; to represent at the national level the concerns of two-year college mathematics educators. *The AMATYC Review* provides an opportunity to publish articles by and for two-year college faculty. A national newsletter is published three times yearly for all members. AMATYC is a member of the National Conference Board of the Mathematical Sciences.

CUT HERE. MAIL TO TREASURER

INDIVIDUAL MEMBERSHIP APPLICATION The American Mathematical Association of Two-Year Colleges

MAIL TO: Karen Sharp, AMATYC Treasurer, Mott Community College, Flint, MI 48502

MEMBER'S NAME _____ POSITION _____

COLLEGE _____

COLLEGE ADDRESS _____ PHONE _____

CITY _____ STATE _____ ZIP _____

RESIDENCE ADDRESS _____ STATE _____ ZIP _____ PHONE _____

Indicate preferred mailing address: ☐ College ☐ Residence

All payments in U.S. funds payable to AMATYC. Membership fee:

☐ $25 Yearly Individual Membership (any person interested in two-year college mathematics)
☐ $10 Yearly Associate Membership (full-time student)

Name of AMATYC Sponsor _____
☐ $22 Reduced Yearly Membership for members of organizations affiliated with AMATYC (see below)
☐ $500 Life Membership

Please check appropriate state affiliation:

☐ AlaMATYC (Alabama)
☐ ArizMATYC (Arizona)
☐ ARKMATYC (Arkansas)
☐ CMC³ (California)
☐ CMC³ South (California)
☐ MATYCONN (Connecticut)
☐ FTYCMA (Florida)
☐ IMACC (Illinois)
☐ IRMC (Indiana)
☐ KYMATYC (Kentucky)
☐ MMATYC (Maryland)
☐ MICHMATYC (Michigan)
☐ MOMATYC (Missouri)
☐ NEMATYC (New England)
☐ MATYCNJ (New Jersey)
☐ OHIOMATYC (Ohio)
☐ ORMATYC (Oregon)
☐ OJCMA (Oklahoma)
☐ PSMATYC (Pennsylvania)
☐ SOCAMATYC (South Carolina)
☐ TMATYC (Tennessee)
☐ TexMATYC (Texas)
☐ VMATYC (Virginia)
☐ WAMATYC (Washington)
☐ WYMATYC (Wyoming)

Membership includes *The AMATYC Review* and *The AMATYC News*. In addition, the following journals are available at extra cost:

☐ Mathematics and Computer Education Journal $16 for 3 issues
☐ The College Mathematics Journal $27 (non MAA members only) for 5 issues
☐ UMPA Journal $25 for 4 issues (U.S. subscriptions only)

Total Amount Enclosed _____ Date Paid _____

American Mathematical Association of Two-Year Colleges

EXECUTIVE BOARD OFFICERS
1987-1989

President
Karl J. Smith
Santa Rosa Junior College
1501 Mendocino Ave.
Santa Rosa, CA 95401
(707) 527-4356

President-Elect
Dale Ewen
Parkland Community College
2400 West Bradley Avenue
Champaign, IL 61821
(217) 351-2225

Treasurer
Karen Sharp
Mott Community College
1401 E. Court St.
Flint, MI 48502
(313) 762-0279 0277

Secretary
Cheryl Cleaves
State Technical Institute
5983 Macon Cove
Memphis, TN 38134
(901) 377-4110

Past President
Stephen B. Rodi
Austin Community College
P.O. Box 2285
Austin, TX 78768
(512) 495-7222

REGIONAL VICE PRESIDENTS

Northeast
Allen R. Angel
Monroe Community College
Rochester, NY 14623
(716) 424-5400 Ext. 5215

Southeast
Margie Hobbs
State Technical Institute
5983 Macon Cove
Memphis, TN 38134
(901) 377-4110

Midwest
James Hall
Parkland Community College
2400 West Bradley Avenue
Champaign, IL 61821
(217) 351-2225

Mid-Atlantic
James Newsom
Tidewater Community College
1700 College Crescent
Virginia Beach, VA 23456
(703) 427-7100

Central
Ron Beeler
East Central Jr. College
Box 529
Union, MO 63084
(314) 583-5193

Southwest
Marilyn Mays
North Lake College
5001 MacArthur Blvd.
Irving, TX 75038-3899
(214) 659-5328

Northwest
Barbara Poole
North Seattle Community College
Seattle, WA 98103
(206) 527-3600

West
Wanda Garner
Cabrillo College
6500 Sequel Drive
Aptos, CA 95003
(408) 688-2075

ACADEMIC COMMITTEES

Developmental Mathematics
Bob Maleda
Allegheny County CC South
West Mifflin, PA 15122

Academic Computer
Wade Ellis
West Valley College
Saratoga, CA 95070

Student Mathematics League
Steve Blasberg
West Valley College
Saratoga, CA 95070

Equal Opportunity
Jean Smith
Middlesex Community College
Middletown, CT 06457

Technical Mathematics
Ray Collings
Tri County Technical College
Pendleton, SC 29670

Education
Geoffrey Akst
Borough of Manhattan C.C.
New York, NY 10007

The lion's share of quality mathematics texts is from-

The Prindle, Weber & Schmidt Series in Mathematics

New for 1989

ARITHMETIC AND ALGEBRA, Second Edition
Rosanne Proga

ELEMENTARY ALGEBRA FOR COLLEGE STUDENTS, Third Edition
Jerome E. Kaufmann

INTERMEDIATE ALGEBRA FOR COLLEGE STUDENTS, Third Edition
Jerome E. Kaufmann

ALGEBRA FOR COLLEGE STUDENTS, Third Edition
Jerome E. Kaufmann

ALGEBRA WITH TRIGONOMETRY FOR COLLEGE STUDENTS, Second Edition
Jerome E. Kaufmann

COLLEGE ALGEBRA WITH APPLICATIONS, Second Edition
James W. Hall and Richard D. Bennett, both of Parkland College

PLANE TRIGONOMETRY, Fifth Edition
Bernard J. Rice and Jerry D. Strange, both of University of Dayton

FUNDAMENTALS OF COLLEGE ALGEBRA, Seventh Edition
Earl W. Swokowski, Marquette University

FUNDAMENTALS OF TRIGONOMETRY, Seventh Edition
Earl W. Swokowski, Marquette University

FUNDAMENTALS OF ALGEBRA AND TRIGONOMETRY, Seventh Edition
Earl W. Swokowski, Marquette University

ALGEBRA AND TRIGONOMETRY WITH ANALYTIC GEOMETRY, Seventh Edition
Earl W. Swokowski, Marquette University

ELEMENTARY LINEAR ALGEBRA, Third Edition
Stewart Venit and Wayne Bishop, both of California State University at Los Angeles

A FIRST COURSE IN DIFFERENTIAL EQUATIONS WITH APPLICATIONS, Fourth Edition
Dennis G. Zill, Loyola Marymount University

DIFFERENTIAL EQUATIONS WITH BOUNDARY-VALUE PROBLEMS, Second Edition
Dennis G. Zill, Loyola Marymount University

NUMERICAL ANALYSIS, Fourth Edition
Richard L. Burden and J. Douglas Faires, both of Youngstown State University

INTRODUCTORY REAL ANALYSIS
James R. Kirkwood, Sweet Briar College

PWS-KENT Publishing Company
20 Park Plaza
Boston, Ma. 02116
A Division of Wadsworth, Inc.
Partners in Education
1-800-343-2204 (in Mass. 1-617-542-3377)

WADSWORTH, INC.– WINNER OF THE NATIONAL ASSOCIATION OF COLLEGE STORES
"Publisher of the Year"
Award for 1988!

AMATYC
SUNY Ag-Tech College
Cobleskill, NY 12043

AMATYC
14th ANNUAL CONVENTION

October 27-30, 1988
Palliser Hotel
Calgary, Alberta, Canada

Convention Co-Chairs:

Steven Terry
Ricks College
Rexburg, ID 83440
U.S.A.
(208) 356-1406

Shao Mah
Red Deer College
Red Deer, Alberta
Canada, T4N 5H5
(403) 342-3300

Program Chair:

Jerine Ridgeway
South Seattle Community College
6000 - 16th Ave. S.W.
Seattle, WA 98106
U.S.A.
(206) 764-5371

Presider Chair:

Barbara Gale
Prince George's College
301 Largo Road
Largo, MD 20772
U.S.A.
(301) 322-0447

BEST COPY AVAILABLE

93

ISSN 0740-8404

ERIC CLEARINGHOUSE FOR
JUNIOR COLLEGES
DEC 2 1988